Report - Fall 2014

Raghav Govind Jha
Department of Physics
Syracuse University
Syracuse, NY

December 15, 2014

Contents

1 Introduction

The aim of studying non-perturbative lattice formulation and constructing lattice actions which possesses subset of supersymmetries of the continuum theory having a Poincare invariant continuum limit has underwent major research over the years.

In 1967, Coleman and Mandula (CM) proved a theorem which was possibly the final nail in the discussion of how the Poincare group and Internal symmetries can be mixed together. They showed that under some assumptions, there was no non-trivial way of combining the two. In a way, it meant that there cannot be any manner in which the fermions can be brought on same footing as bosons. But, then, physics thrives on crisis and exceptions. Few years later, Haag et. al [1] showed that if some of the assumptions in the Coleman-Mandula theorem are relaxed then it is possible to mix these symmetries. The hint was towards the use of graded Lie algebra instead of the normal Lie algebra which was used in the former paper.

We know that any continuous symmetry transformation can be expressed in terms of Lie algebra of linearly independent symmetry generators T_a that satisfy $[T_a, T_b] = if_{abc}T_c$. In much the same way supersymmetry is expressed in terms of symmetry generators T_a that form graded Lie algebra,

$$[T_a, T_b] = T_aT_b - (-1)^{\eta_a\eta_b}T_bT_a = if_{abc}T_c \tag{1}$$

1.1 Lorentz Group & Poincare Group

The 2×2 complex matrices with unit determinant form a group, known as SL(2,C). The letters, S & L stands for special (unit determinant) and linear. 2 denotes the dimensionality and C denotes that these are complex matrices. The group elements depend on 3 complex and 6 real parameters like the Lorentz group. But, they are not the same. This can be checked by noting that if λ is a matrix in SL(2,C), then so is $-\lambda$. They however, produce the same Lorentz transformation. This situation is similar to what happens in case of SO(3) and SU(2). SL(2,C) is a double cover of SO(1,3). We have to resolve this by writing,

$$SO(1,3) \approx \frac{SL(2,C)}{\mathcal{Z}_2}$$

SO(3) is connected since any two points of the parameter domain can be connected by a given continuous path. Both not all the paths can be shrunk to a point. Imagine two antipodal points (North pole and South pole), you cannot shrink a continuous path that connects these two to a given point. Hence, it is not simply connected but only connected. There are two classes of closed paths that are distinct. It is doubly connected. Since SO(3) is not simply connected, it is possible to find a universal covering group for it.

Poincare group is composed of transformations of the form :

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

We talk about Lorentz transformations, if in the above transformation we do not have the a^μ part. Hence, it is pretty obvious to imagine Poincare transformations as direct product between Lorentz transformations and group of 4-translations. In fact, it is not direct but semi-direct product of two.

The Poincare algebra is given by :

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}\right)$$

$$[P^\mu, P^\nu] = 0$$

$$[M^{\mu\nu}, P^\sigma] = i\left(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma}\right)$$

Here, **M** are the anti-symmetric generators of the Lorentz group and **P** are the translation generators.

2 Extending the algebra

The CM theorem clearly meant that there was no non-trivial way of mixing particles with integer and half-integer spin. Wess and Zumino discovered field theoretic models with this extended symmetry (called 'supersymmetry') which connects Bose and Fermi fields and are generated by charge transforming like spinors under Lorentz group (supercharges). These supercharges give rise to a new system of commutation and anti-commutation relations, which is not precisely a Lie algebra but a graded algebra. This has a $\mathcal{Z}_2$ grading. In 1975, Haag, Lopuszanski & Sohnius showed that the energy-momentum operators appear among the elements of this pseudo Lie algebra which hints that the fusion between internal and space-time symmetries must exist.

The Poincaré generators P^μ and $M^{\mu\nu}$ are bosonic generators. In supersymmetry, we add fermionic generators Q^L_α, $\bar{Q^M_\beta}$, where L, M = 1,2,...$\mathcal{N}$. The $\mathcal{N} = 1$ case is simple supersymmetry and $\mathcal{N} > 1$ is extended supersymmetry.

The complex spinorial generators follow the following algebra :

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} Z^{LM}$$

$$[P, Q] = 0$$

$$[Q^L_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^L_\beta$$

$$\{Q^L_\alpha, \bar{Q^M_\beta}\} = \delta^{LM}\sigma^\mu_{\alpha\beta} P_\mu$$

The last one is the most interesting of these four. It roughly means that the supersymmetric generators are square root of the four-momentum. It also means that combining two supersymmetric transformations (one of each helicity) corresponds to space-time translation. Also, in our discussion we neglect any central charges denoted by Z in the first equation. That then reduces to,

$$\{Q^L_\alpha, Q^M_\beta\} = 0$$

3 SUSY Algebra on a Lattice - Methods

Once we have the required algebra, we can immediately ask how to put this algebra on a lattice. This is a difficult task. A look at the last of the anti commutation relation of supercharges tells us that it is impossible. There is no notion of translations on a lattice. Supersymmetry cannot be an exact symmetry on a lattice.

3.1 Supercharges, R-Symmetries

The SUSY algebra is highly constrained and there only exist only some possibilities for the number of supercharges. The theory should not contain spin greater than 1 since this should be the case for a gravity-free theory yet renormalizable. Like $\mathcal{N} = 1$ supersymmetry for d = 4 has $\mathcal{Q} = 4$ and a theory with $\mathcal{N} = 4$ supersymmetry for d = 4 has $\mathcal{Q} = 16$. Note that $\mathcal{Q}$ is the number of real supercharges. The SUSY generators come with their own internal symmetry called as R-symmetry. More precisely, this is the global chiral symmetry which do not commute with the supercharges (because there is no analogous phase rotation of the gluino's partner, gluon). Also, bosonic and fermionic fields along with Eucliean Lorentz symmetry furnishes a representation of the R-symmetry. The $\mathcal{Q} = 4, d = 4$ SYM theory has a U(1) R-symmetry. Also dimensional reduction of the supersymmetric theories leads to enlargement of R-symmetry group by means of Euclidean Lorentz generators acting in the reduced dimensions. For ex : the $\mathcal{N} = 1$ theory in $d' = 10$ dimensions dimensionally reduced to d dimensions has a SO(d) Lorentz symmetry and SO(10-d) R-symmetry. Also, it is important to note that presence of central charges i.e Z affect the R-symmetry. If the central charges all vanish, then the R-symmetry group is $U(\mathcal{N})$ or else it is a subset of this.

3.2 Two Methods : Implementing SUSY on Lattice

The first method is related to the idea of twisting and Dirac-Kaehler fermions. This involves decomposition of Lorentz spinor supercharges into a sum of integer p-form tensors under a diagonal subgroup of Lorentz group and some large global chiral symmetry known as R-symmetry. The zero form supercharge that comes out of twisting is nilpotent (does not generate any translation , also like BRST charge) and constitutes a closed subalgebra of the full twisted superalgebra. This supersymmetry can be made manifest in the lattice action because there is no notion of translation involved [2]. The second method is that of contructions based on the ideas of deconstruction and orbifolding. The orbifold technique is a powerful way of generating all known SYM lattices. The starting point is the deconstruction method of Arkani-Hamed, Cohen and Georgi (AHCG) [**?**]. This is a model that latticizes supersymmetry without intending to do so. This model can be retrieved through the process of orbifolding. We will revisit these two methods in some detail later.

4 Supersymmetry Theories & Lattice Constructions

It is clear that entire continuum supersymmetry cannot be implemented on a lattice rightaway. One practical approach is to construct a lattice theory that respects as many symmetries of the continuum target theory as possible and limiting the number of operators which need fine-tuning. Even if we cannot realize all of the symmetries of the target theory, we can hope that it will emerge as 'accidental symmetries'. An accidental symmetry is a symmetry that emerges in the IR limit of the theory even if it was not respected by full lattice action. Baryon number violation in GUT is an example of this. It is infact, this, accidental symmetry that restores Poincare symmetry and even supersymmetry from the lattice action. Let's see a minimal case where this looks possible.

4.1 $\mathcal{N} = 1$ supersymmetry with d=4

$$\mathcal{L} = \bar{\lambda} i \bar{\sigma}^m D_m \lambda - \frac{1}{4} V_{mn} V^{mn}$$

where $\bar{\sigma}^m = (1, -\sigma)$ and V_{mn} is the gauge field strength.

Now we need to ask : What terms can be added to this Lagrangian such that it breaks SUSY but is consistent with gauge & Lorentz symmetry ?

We can add a gaugino (super partner of gauge fields) mass term.

$$\delta\mathcal{L} = m\lambda\lambda + h.c$$

This terms breaks the supersymmetry and the R-symmetry. Therefore, if we can ensure the R-symmetry, then it will forbid the gaugino mass term and the theory will be accidentally SUSY.

4.2 Supersymmetric QM on a lattice - A Toy Model

This is a good toy model to help the understanding and also realizing the problems encountered while trying to study SUSY on lattice. Witten gave a continuum theory which comprises of single commuting bosonic coordinate and two anti commuting fermonic coordinate. This is just 0+1-dim SUSY.

$$S = \int dt \frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + \frac{1}{2} P'(\phi)^2 + \psi_i \frac{d\psi_i}{dt} + i\psi_1\psi_2 P''(\phi) \tag{2}$$

Here, $P'(\phi)$ is some polynomial in ϕ and P'' is its derivative. $P(\phi)$ is called super potential.

The above action is invariant under two super symmetries given below where ϵ_A and ϵ_B are infinitesimal Grassmann parameters (only writing for ϵ_A here).

$$\delta_A \phi = \psi_1 \epsilon_A$$

$$\delta_A \psi_1 = \frac{d\phi}{dt} \epsilon_B$$

$$\delta_A \psi_2 = i P' \epsilon_A$$

Using these infinitesimal changes in the action above, we get :

$$\delta_A S = \int dt \;\; i\epsilon \left(P' \frac{d\psi_2}{dt} + \frac{d\phi}{dt} P'' \psi_2 \right)$$

Integration by parts the above equation, sets the term in bracket to be zero. But, this is where the trouble is when we adopt a lattice version of the theory which we discuss in Section 4.2.1. The notion of Leibnitz rule is not valid for lattice difference operators.

4.2.1 Naive discretization

We now discretize the above theory in a simple manner. Define the field on lattice sites $x = na$, $n = 0, 1....L-1$ and replace the integrals by sum using the periodic boundary conditions. The periodic boundary conditions brings us to the problem of fermion doubling, which can be avoided by replacing the continuum derivative with forward (backward) difference operators given by :

$$\Delta^+ f_x = f(x+a) - f(x)$$

Using this and carrying out the supersymmetric variation, one finds a non vanishing variation in action as :

$$\delta_A S_L = \sum_x i\epsilon \;\; \left(P' \Delta^- \psi_2 + \Delta^- \phi P'' \psi_2 \right)$$

Using the rule of lattice integration by parts [1] we find,

$$\delta_A S_L = i \sum_x \epsilon \psi_2 \left(-\Delta^+ P' + \Delta^- \phi P'' \right)$$

This term vanishes in the continuum but clearly does not for any finite lattice spacing. Thus, the naive lattice action is not invariant under supersymmetric transformations. This problem can be avoided and the lattice action can be made to be invariant under supersymmetry if we also consider (the B part of supersymmetry) and take a linear combination of both of them. The derived supersymmetry will no longer be the square root of translation, but will be nilpotent instead. Also important to note is that two supersymmetries were required to find such a nilpotentsupercharge ? the continuum theory has extended supersymmetry. This will be later seen as a general property of lattice models with exact supersymmetry.

4.2.2 Nicolai Map

In theories with a global supersymmetry there exists a mapping (generally, non-local) of the bosonic fields whose determinant cancels the Pfaffian (Salam-Mathews determinant) of the fermionic fields present. This existence of the 'Nicolai Map' is central to the idea of implementing models in a SUSY preserving way on lattice. In fact, as shown in [16] it is also possible to formulate supersymmetry on a discrete space time lattice by preserving Nicolai map as substitute to SUSY algebra. Let us consider the SUSY QM Lagrangian :

$$L = \frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 + \frac{1}{2} P'(\phi)^2 + \psi_i \frac{d\psi_i}{dt} + i\psi_1 \psi_2 P''(\phi)$$

[1] $\sum_x f(x) \Delta^- g(x) = -\sum_x g(x) \Delta^+ f(x)$

where $P(\phi)$ is a super potential.
If we consider the mapping (called 'Nicolai mapping') from ϕ to ξ as ,

$$\xi = \frac{d\phi}{dt} + P'(\phi)$$

,

We observe that the Jacobian of this change of variables i.e $\frac{\delta\xi}{\delta\phi}$ exactly cancels the fermionic determinant and thus the effective lagrangian for ξ becomes gaussian except a total derivative term that can be neglected owing to periodic boundary conditions. The partition function then takes the form,

$$Z = \int \mathcal{D}\xi \ e^{-\sum_x \xi^2} \tag{3}$$

This has an immediate advantage. The form of the super potential has disappeared from 'Z' and hence it cannot depend on any coupling constants in the model. It is *topologically invariant.*

4.3 $\mathcal{N} = 2, 2$ SYM

The twisted formulation of a given continuum field theory is a necessary condition for constructing a model with exact lattice supersymmetry but it is not sufficient. Among other requirements, are, maintaining exact gauge symmetries when we discretize and work on lattice. The simplest model we will study here in $\mathcal{N} = 2, 2$ SYM.

The study was first initiated using the orbifold technique and then the twisted constructions were also implemented. The latter approach starts with a twisted formulation of the continuum theory action given by :

$$S = \frac{1}{2g_0^2} Q \ \mathrm{Tr} \int d^2x \left[\frac{1}{4}\eta[\phi, \bar{\phi}] + 2\chi_{12}F_{12} + \chi_{12}B_{12} + \psi_\mu D_\mu \psi \right]$$

Here, all the fields are in their usual adjoint representation. (See precise mathematical reason). One important note is that the dimension is equal to the number of generators.

The covariant derivatives act as :

$$D_\mu f = \partial_\mu f + [A_\mu, f]$$

while the action of Q on twisted fields are given by :

$$\begin{aligned} QA_\mu &= \psi_\mu \\ Q_{\psi_\mu} &= -D_\mu \phi \\ Q\bar{\phi} &= \eta \\ Q\eta &= [\phi, \bar{\phi}] \\ QB_{12} &= [\phi, \chi] \\ Q\chi_{12} &= B_{12} \\ Q\phi &= 0 \end{aligned} \tag{4}$$

We note that $Q^2 = \delta_\phi$ [2]
carrying out the Q variation of the action and integrating over the auxillary field B_{12} gives :

$$S = \frac{1}{2g_0^2}\mathrm{Tr}\int d^2x \left\{ \frac{1}{4}[\phi,\bar{\phi}]^2 - \frac{1}{4}\eta[\phi,\eta] - F_{12}^2 - D_\mu\phi D^\mu\bar{\phi} - \chi_{12}[\phi,\chi_{12}] \right.$$
$$\left. -2\chi_{12}(D_1\psi_2 - D_2\psi_1) - \psi_\mu D_\mu\eta + \psi_\nu[\bar{\phi},\psi_\mu] \right\}$$

4.3.1 Twisted $\mathcal{N}$ = 2, d = 2 SYM

We will review the twisted construction of the $\mathcal{N} = 2$ SYM in two dimensions. This theory can be obtained by the dimensional reduction of $\mathcal{N} = 1$ SYM theory in four dimensions. The global symmetry of the four dimensional theory, $SO(4)_E \times U(1)$ where $SO(4)$ is the Euclidean Lorentz symmetry and the $U(1)$ part is the chiral symmetry splits through dimensional reduction to become the global symmetry of the above mentioned two dimensional theory as ,

$$SO(4)_E \times U(1) \rightarrow SO(2)_E \times SO(2)_R \times U(1)$$

Here, $SO(2)_E$ is the Euclidean Lorentz symmetry ; $SO(2)_{R1}$ is the rotational symmetry among the dimension which was reduced and $U(1)_{R2}$ is the chiral U(1) symmetry of the theory. We can rewrite it in following manner :

$$SO(4)_E \times U(1) \approx SO(2)_E \times SO(2)_{R1} \times SO(2)_{R2}$$

We see that the internal symmetry group has two SO(2)'s and that means that there are two possible twists we can have. They are called the A-twist and B-twist. In the A-twist, the twisted rotation SO(2) is defined as the diagonal SO(2) subgroup of the product of $SO(2)_E$ and $SO(2)_{R2}$ (chiral) symmetry. In the B-twist, the twisted rotation is defined as the diagonal SO(2) subgroup of the product of $SO(2)_E$ and $SO(2)_{R1}$ (internal) symmetry. The B-twist is also known as *self-dual twist* . We can combine the scalars and gauge fields to get a complexified gauge field in the *self-dual twist* written as ,

$$\mathcal{A} = A_1 + iA_2 \quad ; \quad \bar{\mathcal{A}} = A_1 - iA_2$$

4.3.2 Lattice theory for $\mathcal{N} = (2,2)$ SYM

We will discuss the discretization of the *self-dual twist* of the two-dimensional Yang-Mills model with $\mathcal{Q} = 4$ supercharges. A geometrical scheme was proposed by Catterall [9]. The continuum p-form fields are mapped to lattice fields defined on *p-subsimplices* of a general lattice. In case of hypercubic lattices, this assignment is similar to placing a p-form with indices $\mu_1, \mu_2 .... \mu_p$ on the link connecting $\mathbf{x}$ with $(\mathbf{x} + \mu_1 + ... + \mu_p)$ where the μ_i, i = 1,....p, corresponds to the unit lattice vector. Also, each possible link has two possible orientations. A *positively* oriented field corresponds to one which has positive components with respect to the coordinate basis. The

[2] $Q^2 A_\mu = -D_\mu\phi$
$Q^2 A_\mu = -\partial_\mu\phi - [A_\mu, \phi]$

continuum derivative on hypercubic lattice are represented by lattice difference operators acting on the link fields. To be exact, the covariant derivatives (appearing in curl like operations) acting on positively oriented fields are replaced by lattice gauge covariant forward difference operators whose action on scalar and vector are given below :

$$\mathcal{D}^{+}_{\mu} f(x) = \mathcal{U}_{\mu}(\mathbf{x}) f(\mathbf{x}+\mu) - f(\mathbf{x})\mathcal{U}_{\mu}(\mathbf{x}) \tag{5}$$

$$\mathcal{D}^{+}_{\mu} f_{\nu}(x) = \mathcal{U}_{\mu}(\mathbf{x}) f_{\nu}(\mathbf{x}+\mu) - f_{\nu}(\mathbf{x})\mathcal{U}_{\mu}(\mathbf{x}+\nu) \tag{6}$$

where $\mathbf{x}$ denotes a two dimensional lattice vector and $\mu = (1,0)$, $\nu = (0,1)$ unite vectors in different directions. We have replaced the continuum complex gauge fields $\mathcal{A}_{\mu}$ by non-unitary link fields $\mathcal{U}_{\mu} = e^{i\mathcal{A}_{\nu}}$. The backward difference operator $\bar{\mathcal{D}}^{-}_{\mu}$ replaces the continuum covariant derivative (appearing in divergence like operations) acting on positively oriented lattice vector fields in following way :

$$\bar{\mathcal{D}}^{-}_{\mu} f_{\nu}(x) = f_{\mu}(\mathbf{x})\bar{\mathcal{U}}_{\mu}(\mathbf{x}) - \bar{\mathcal{U}}_{\mu}(\mathbf{x}-\mu) f_{\mu}(\mathbf{x}-\mu) \tag{7}$$

5 Orbifolding Projection

In the famous paper by AHCG [12] , they tried to construct a fifth dimension of the base 4-theory which was needed to account for the many ‘ flavors ’ of 4-theory. In order to avoid some difficulties they took four continuous dimensions and a latticized fifth dimension. In the moose diagram below, the open circles are gauge group $U(k)$ and it is a N-sided polygon. The matter (fermionic) fields are in form of chiral supermultiplets which appear as the link connecting the two nodes and transform as bilinears. The mechanism by which enhanced supersymmetry emerges in the continuum limit of the AHCG model is what was always sought for in the lattice constructions. But since the AHCG model is a theory in four continuous dimensions, we need to follow a reverse approach to find general principles of how it is constructed and then apply them to construct true space time lattices.

The lattices obtained by orbifolding projection of "mother theory" has same supersymmetry as the target theory. The projection makes it possible for a subset of supersymmetries to be exactly put on lattice and also protecting the theory from unwanted relevant operators in continuum limit.

The group $G_R = SU(2) \times SU(2) \times U(1)$ has seven generators (3+3+1). We can denote them by L_a, R_a & Y respectively where Y is the single generator of U(1) and others are SU(2) generators.

The integer charges $\mathbf{r} = r_1, r_2$ are constructed from Cartan sub-algebra. They are related by :

$$r_1 = -L_3 + R_3 - Y \quad , \quad r_2 = L_3 + R_3 - Y \tag{8}$$

6 Topological Field Theories (TFT)

One of the easiest ways of making a topological field theory is to take a extended space-time supersymmetric theory and *twist* it. A common feature of both supersymmetric lattice theories and topological field theories is the presence of nilpotent scalar supercharge $\mathcal{Q}$. They have actions which are Q-exact. All the supersymmetric lattice theories are associated with topological field theories but the opposite is not always true. Given a supersymmetric twist, we are not guaranteed of having a well-defined lattice theory.

TFT are characterized by the fact that the energy momentum tensor vanishes. It can be of two types : 1. Schwarz type 2. Cohomological type (Witten-type).

6.1 Schwarz Type TFT's

In this type of TFT's, the action is explicitly independent of the metric. Two examples of this type of theories are 1) Chern-Simons (CS) theory & 2) BF (Background field) Model. The metric independence implies that the stress-energy tensor of TFT vanishes . $\frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu} = 0$. There are no propagation local degrees of freedom ; only degrees of freedom are topological.

6.2 Witten Type TFT's

In Witten-type topological field theories, the topological invariance is more subtle. The lagrangian generally depend on metric explicitly, but one shows that the expectation value of the partition function and special classes of correlation functions are diffeomorphism [3] invariant.

7 $\mathcal{N} = 4$ SYM in $d = 4$

The most famous among all SYM theories is $\mathcal{N} = 4$ SYM in $d = 4$. It has a coupling constant which does not run, and is conformal. It can be thought of as the most symmetric theory in four dimensions without gravity. To obtain the target $\mathcal{N} = 4$ theory in four dimensions, we start with $\mathcal{Q} = 16$ matrix model. The matrix model can be obtained by dimensionally reducing the d=10 $\mathcal{N} = 1$ SYM down to zero dimensions. The reduced model possesses SO(10) R-symmetry inherited from the Lorentz symmetry of the d=10 dimensional theory before reduction. The field content of the mother theory is ten bosonic and sixteen fermionic matrices transforming as **10** and **16** representations of R-symmetry in the adjoint rep. of gauge group. This theory also has an S-duality under which :

$$\tau_{YM} = \frac{\theta}{2\pi} + i\frac{2\pi}{g_{YM}^2}$$

goes to $1/\tau$. Also, we define the 't Hooft coupling $\lambda = g_{YM}^2 N$.

The field content of $\mathcal{N} = 4$ SYM are the two Spin-1 gauge bosons $\mathcal{A}_\mu$, six massless scalar fields (Spin-0) Φ^I, I = 1...6, four chiral fermions ψ_α^a and four anti chiral fermions $\psi_{\dot\alpha}^a$ with a = 1...4. The indices α and $\dot\alpha = 1..2$ are the spinor indices of SU(2) that make up the 4-d Lorentz algebra. All fields transform under the adjoint representation of SU(N) gauge group.

$$16 = 1 \bigoplus 4 \bigoplus 6 \bigoplus 4 \bigoplus 1 \tag{9}$$

where, Q_a , $a = 1....4$ shifts the helicity by 1/2. For any SU(N) gauge group, the one-loop β function for the gauge coupling g_{YM} is given by [15]

[3]Roughly speaking, this means that they are metric independent

$$\beta_1(g_{YM}) = \mu\frac{\partial g_{YM}}{\partial\mu} = \frac{-g_{YM}^3}{16\pi^2}\left(\frac{11N}{3} - \frac{1}{6}\sum_i C_i - \frac{1}{3}\sum_j C_j\right) \tag{10}$$

where, the first sum is over the real scalars with quadratic Casimir and second is over all Weyl fermions with quadratic Casimir. The vanishing beta function at one loop for $\mathcal{N}=4$ SYM is a very important property and can be seen directly from Eq.(4) if we note the fact that we have 6N scalars and 8N Weyl fermions corresponding to the gauge group SU(N) under consideration for $\mathcal{N}=4$ SYM.

We decompose the variables of the mother theory under the SU(5) $\bigotimes$ U(1) subgroup of SO(10) as :

Bosons : $\mathbf{10} \rightarrow 5 \bigoplus \bar{5} = z^m \bigoplus \bar{z}_m$
Fermions : $\mathbf{16} \rightarrow 1 \bigoplus 5 \bigoplus \bar{10} = \lambda \bigoplus \psi^m \bigoplus \zeta_{mn}$

7.1 Hypercubic Lattice

Various fields of the SU(5) multiplets distribute to the hypercubic lattice as λ (0-cell), ψ^m (0-cell & 4-cell), ζ_{mn} (2-cell, 3-cell). Thus the fermions are totally antisymmetric p-cell variables. This provides explicit realization of the Dirac-Kahler fermions. The distributions of bosons and their orientations are governed by the fermions because of the exact supersymmetry.

7.2 The Twisted Construction

There are three inequaivalent twists of the $\mathcal{N}=4$ SYM theory in four dimensions partly due to Yamron, Vafa & Witten & Marcus. Only the last one of these correspond to the orbifold lattice construction and will be implemented here. The $\mathcal{N}=4$ SYM theory in d=4 dimensions possesses a global Euclidean Lorentz symmetry $SO(4)_E \sim SU(2) \times SU(2)$ on $\mathcal{R}^4$ and a global R-symmetry group SO(6) or Spin(6) $\sim SU(4)$. The complexification of Spin(4) is SL(2,$\mathbb{C}$) and the two spin representations are (**2**,**1**) & (**1**,**2**) [4]. The spin representations of *Spin(6)* are the four dimensional representation **4** of SU(4) and its dual $\bar{\mathbf{4}}$. This means that the four-dimensional fermionic fields transform under :

$$\text{Spin}(4) \times \text{Spin}(6) \sim \text{SL}(2,\mathbb{C}) \times \text{Spin}(6)$$

as,

$$(\mathbf{2}, \mathbf{1}, \bar{\mathbf{4}}) \bigoplus (\mathbf{2}, \mathbf{1}, \mathbf{4})$$

[4] This notation can be slightly off putting, many authors prefer (**1/2**,**0**) & (**0**,**1/2**) instead.

Theory	R-symmetry group	Orbifolding	Maximal Twist
$d=2, \mathcal{Q}=4, \mathcal{N}=2$	$SO(2) \bigotimes U(1)$	Yes	Yes
$d=2, \mathcal{Q}=8, \mathcal{N}=4$	$SO(4) \bigotimes SU(2)$	Yes	Yes
$d=2, \mathcal{Q}=16, \mathcal{N}=8$	$SO(8)$	Yes	Yes
$d=3, \mathcal{Q}=4, \mathcal{N}=1$	$U(1)$	No	No
$d=3, \mathcal{Q}=8, \mathcal{N}=2$	$SO(3) \bigotimes SU(2)$	Yes	Yes
$d=3, \mathcal{Q}=16, \mathcal{N}=4$	$SO(7)$	Yes	Yes
$d=4, \mathcal{Q}=4, \mathcal{N}=1$	$U(1)$	No	No
$d=4, \mathcal{Q}=8, \mathcal{N}=2$	$SO(2) \bigotimes SU(2)$	No	No
$d=4, \mathcal{Q}=16, \mathcal{N}=4$	$SO(6)$	Yes	Yes

The important inference from the table is that there exists a parallel between the theories we can either twist or orbifold.

7.3 The A_4^* Lattice

The symmetry of the hypercubic lattice action is S_4, which is smaller than the symmetry of the hypercube. There exists a more symmetric lattice than the hypercubic lattice for the d=4, $\mathcal{N}=4$ theory. This lattice is called A_4^* lattice. Infact, A_4 lattice is generated by simple roots of SU(5) = A_4 ; then A_4^* is the dual lattice generated by fundamental weights of SU(5) (or in other words, defining representation of SU(5)). Lower dimensional analogs are the triangular lattice (A_2^*). On this lattice, all the five basis vectors are treated equally and they are oriented in such a way that the basis vectors connect the center of 4-simplex (i.e Pentachoron) to its corners. This greater symmetry is advantageous, more symmetric lattice means lesser relevant operators on the lattice. The lattice possesses S^5 point group symmetry (Weyl group of SU(5)). This point symmetry group has 120 elements (5!) and seven conjugacy classes.

8 Conformal Field Theory : An Introduction

The transformations which locally preserve the angle between any two lines is said to be conformal. It is defined mathematically as below :

Let there be differentiable maps $\phi : U \mapsto V$, where U $\in$ M and V $\in$ M' are open subsets. A map ϕ is conformal, if the metric tensor satisfies $\phi^* g' = \Lambda g$. Denoting $x' = \phi(x)$ with $x \in U$, it becomes

$$g'_{\rho\sigma}(x') \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x)$$

We note that $\Lambda = 1$ gives us the usual Poincare transformation. CFT have SO(D+1,1) symmetry group.

Conformal Transformations is an extended symmetry group consisting of 10 Poincare, 4 Special conformal symmetry and one dilatation. The generator of dilatations D plays a very important role in the quantum structure of $\mathcal{N}=4$ SYM. The Poincare subgroup of conformal group does not admit any quantum corrections, but the dilatation generator D does inspite of the conformal nature of theory. Generally, to find the anomalous dimension of an operator $\mathcal{O}(x)$, one considers its two-point correlation with itself given by :

$$\mathcal{O}(x)\bar{\mathcal{O}}(y) \approx \frac{1}{|x-y|^{2\Delta}}$$

Conformal symmetry is a very peculiar and constraining property of a field theory. As compared to QFT's in four dimensions, conforal field theories can be defined in a rather abstract way via operator algebra and representations. There are also instances where the usual description in terms of a Lagrangian action is not even known.

The method of calculating the correlation functions by simply assuming the crossing symmetry is known as *bootstrap approach.* (P 186 Di Fran)

Importantly, the one point correlation function vanishes due to conformal symmetry and the two-point and three-point functions are completely determined upto some factor (called structure constants) by the scaling dimensions (given above).

The physical observables of a gauge group are its gauge invariant operators.

- Translation : $\widetilde{x}^\mu = x^\mu + a^\mu$
- Dilation : $\widetilde{x}^\mu = \alpha x^\mu$
- Rotation : $\widetilde{x}^\mu = M^\mu_\nu x^\nu$
- SCT : TODO!!

One famous real life example of CFT's is the second-order phase transition. At the critical point, the correlation length diverges and the scale invariance at the point also implies the conformal invariance.

A Review of Group Theory

A group, $\mathcal{G}$ is a set with rule for assigning to every ordered pair of elements, a third element such that following holds :

- $\forall$ f,g $\in \mathcal{G}$ $\exists$ h such that $h = f \bullet$ g $\in \mathcal{G}$
- $\forall$ f,g,h $\in \mathcal{G}$, f $\bullet$ (g$\bullet$ h) = (f $\bullet$ g) $\bullet$ h
- There is an identity element, e, such that $\forall$ f $\in \mathcal{G}$, e $\bullet$ f = f $\bullet$ e = f
- Every element $f \in \mathcal{G}$ has an inverse f^{-1} such that $f^{-1}\bullet$ f = f $\bullet$ f^{-1} = e

In addition to this, a Lie group (real) is a smooth manifold G together with the group structure such that both the multiplication and inverse are smooth maps. *Smooth* means infinitely differentiable. Complex Lie group is a complex manifold with same structure but holomorphic maps. [5]. SU(3) is a Lie group with eight parameters. It means that associated Lie algebra must have eight independent generators. The number of simultaneously diagonalizable generators is called the rank of the Lie group. Rank can also be thought of as the number of mutually commuting generators of a Lie group. In general, SU(N) has rank N-1. SO(2N) and SO(2N+1) both have rank N. U(N) has rank N. Clearly, the generators of SO(3) i.e L_x, L_y & L_z don't commute with each other and rank of SO(3) is 1. Also, an operator which commutes with all the generators of a given Lie group is known as Casimir operator. According to a theorem due to Racah, the number of independent Casimir operators of a group is equal to its rank.

B Z_N Symmetry Groups

Z_N group describes a symmetry of a plane figure invariant after a rotations of $2\pi/N$ degrees. In particular, Z_2 is a group of just two elements. Rotations by 180 and 360 degrees bring back the original configuration. The letter Z and S have this symmetry. It is ask clear that the group Z_2 is a subgroup of U(1).

Bifundamental representation is kind of tensor product between two different gauge groups whereas if they are same are known as adjoint representation. Therefore, adjoint representation can be viewed as a bifundamental representation from a gauge group to itself.

First let's consider $M = \mathcal{R}^4$ which has the rotational symmetry Spin(4) [6] ; while the $\mathcal{N} = 4$ theory has the larger symmetry Spin(4) $\bigotimes$ Spin(6). *Twisting* means replacing the Spin(4) by a different subgroup of Spin(4) $\bigwedge$ Spin(6) which we will denote Spin$\acute{(}$4). This new subgroup acts on $\mathcal{R}^4$ in the same way but acts differently on the $\mathcal{N} = 4$ theory.

The $\mathcal{N} = 4$ SYM has number of interesting properties. The beta function vanishes identically (coupling doesn't run).

The fundamental representation is one where the matrices representing the group elements are just themselves, $M(g) = g$. For SU(N) and SO(N), these are just the $N \times N$ matrix. This can also be written as : $X \to M(g)X = gX$. In the adjoint representation however, the action of a group element U on a Lie algebra element T is given by : $T \to UTU^{\dagger}$. It is also worth pointing

[5]A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain

[6]We will use Spin(N) and SO(N) interchangeably, since they are related by double covering

that fields transform as the adjoint representation of the gauge group to ensure that the business of covariant derivatives transforms homogenously.

References

[1] All possible generators of supersymmetries of the S-matrix

[2] S.Catterall, "From Twisted Supersymmetry to Orbifold Lattices", [hep-lat/0712.2532]

[3] S.Catterall, "Monte Carlo renormalization group minimal walking technicolor"

[4] Orbifold and Twisted SUSY

[5] Note on Dirac-Kahler Equation

[6] A lattice formulation of super Yang-Mills theories with exact supersymmetry

[7] Geometry of orbifolded supersymmetric lattice gauge theories

[8] Twisted SUSY and AdS - CFT

[9] S.Catterall, "A geometrical approach to N = 2 super Yang-Mills theory on the two dimensional lattice "

[10] Lattice SUSY via Twisting

[11] Geometrical approach of N=2 SYM

[12] AHCG Model

[13] Resummation and S-duality in N = 4 SYM

[14] The N=4 Superconformal Bootstrap

[15] Gross & Wilczek, "Ultraviolet Behavior of Non-Abelian Gauge Theories"

[16] Sakai & Sokomoto, "Lattice supersymmetry and the nicolai mapping"