24/2c Great.

PHY 661 - Quantum Mechanics I : Assignment 6

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10 October 2013

38. Prove that $(U_q\psi)(q) = \langle q|U_q|\psi\rangle = \langle q|\psi\rangle = \psi(q\text{-}a)$

4/4.

Solution : We know that $U_q(a) = \exp\left(\frac{-i}{\hbar}\vec{\hat{p}}\cdot\vec{a}\right)$ ✓

then $U_q(-a) = \exp\left(\frac{i}{\hbar}\vec{\hat{p}}\cdot\vec{a}\right) = U_q^{-1}(a)$ ✓ - ①

Since this is unitary, $[U_q(\vec{a})]^\dagger = U_q^{-1}(\vec{a})$ ✓ - ②

We have $[\hat{U}_q(\vec{a})]^\dagger = \exp\left(\frac{i}{\hbar}\vec{\hat{p}}\cdot\vec{a}\right)$ ✓

$[UU^\dagger = \mathbb{1}]$ ↓ Hermitian $\Rightarrow U^{-1} = U^\dagger$

Now $\langle\vec{q}|U_q|\psi\rangle = \langle\psi|\hat{U}_q^\dagger|\vec{q}\rangle^*$ ✓ - ③

We also know $U_q(\vec{a})|\vec{q}\rangle = |\vec{q}+\vec{a}\rangle$ ✓ ← Lecture notes & proved in class

$U_q^\dagger(\vec{a})|\vec{q}\rangle = U_q^{-1}(\vec{a})|\vec{q}\rangle$

$= |\vec{q}-\vec{a}\rangle$ ✓ - ④

using ③ & ④

$\langle\vec{q}|U_q|\psi\rangle = \langle\psi|\vec{q}-\vec{a}\rangle^*$ ✓

$= \langle\vec{q}-\vec{a}|\psi\rangle = \psi(\vec{q}-\vec{a})$ ✓

6/6. 41 a. *Prove the Heisenberg's Relation*

$$\Delta q \Delta p \geq \frac{\hbar}{2}$$

Use only Schwarz's Inequality & Commutation Relations

<u>Solution</u> :

$$(\Delta q)^2(\Delta p)^2 = \langle\psi|(q-\langle q\rangle)^2|\psi\rangle\ \langle\psi|(p-\langle p\rangle)^2|\psi\rangle \quad ✓ \tag{2}$$

Let us define $\tilde{q}$ and $\tilde{p}$ as,

$$\tilde{q} = q - \langle q\rangle$$
$$\tilde{p} = p - \langle p\rangle \quad ✓$$

Now, we will show that $\tilde{q}$ and $\tilde{p}$ follow the same commutation relations as p & q .

$$[\tilde{q},\tilde{p}] = (q-\langle q\rangle)(p-\langle p\rangle) - (p-\langle p\rangle)(q-\langle q\rangle)$$

$$= qp - q\langle p\rangle - \langle q\rangle p + \langle q\rangle\langle p\rangle - pq + p\langle q\rangle + \langle p\rangle q - \langle p\rangle\langle q\rangle$$

$$= qp - pq = [q,p] \quad ✓$$

Now, writing (1) in terms of $\tilde{q}$ and $\tilde{p}$.
We get,

$$(\Delta q)^2(\Delta p)^2 = \langle\psi|(\tilde{q})^2|\psi\rangle\ \langle\psi|(\tilde{p})^2|\psi\rangle \tag{3}$$

Since, both $\tilde{q}$ and $\tilde{p}$ and Hermitian we can write it as :

$$(\Delta q)^2(\Delta p)^2 = \langle\tilde{q}\psi|\tilde{q}\psi\rangle\ \langle\tilde{p}\psi|\tilde{p}\psi\rangle \tag{4}$$

Use Schwarz inequality

$$|V_1|^2|V_2|^2 \geqslant |\langle V_1|V_2\rangle|^2 \quad ✓$$

Since $\hat{q}^2 = \hat{q}\hat{q} = \hat{q}^\dagger\hat{q}$ and $\hat{p}^2 = \hat{p}^\dagger p$

we get from (4)

$$(\Delta q)^2(\Delta p)^2 \geqslant |\langle\tilde{q}\psi|\tilde{p}\psi\rangle|^2 \quad - (5)$$

Now, using the fact that $\langle\tilde{p}\psi|\tilde{q}\psi\rangle = \langle\psi|\hat{p}^\dagger q|\psi\rangle = \langle\psi|\hat{p}\hat{q}|\psi\rangle$

and $\langle\tilde{q}\psi|\tilde{p}\psi\rangle = \langle\psi|\hat{q}\hat{p}|\psi\rangle$ ✓

we write (5) as

$$(\Delta q)^2(\Delta p)^2 \geqslant |\langle\psi|\hat{q}\hat{p}|\psi\rangle|^2$$

Turn over.

we decompose $\hat{q}\hat{p}$ now as

$$\hat{q}\hat{p} = \frac{1}{2}[\hat{q},\hat{p}] + \frac{1}{2}\{\hat{q},\hat{p}\}$$ ✓

where $\{\cdot,\cdot\}$ stands for anti-commutator

i.e $\{A,B\} = AB+BA$

$$(\Delta q)^2 (\Delta p)^2 \geq \left| \left\langle \psi \left| \frac{1}{2}\{\hat{q},\hat{p}\} + \frac{1}{2}[\tilde{q},\tilde{p}] \right| \psi \right\rangle \right|^2$$

$[\tilde{q},\tilde{p}] = i\hbar$

$$\geq \frac{1}{4}\left|\langle\psi|\{\hat{q},\hat{p}\}|\psi\rangle + \frac{i}{4}\langle\psi|\hbar|\psi\rangle\right|^2$$ ✓

$\{\tilde{q},\tilde{p}\}$ is hermitian and has real ~~eigenvalue~~ expectation value.

$|a+ib|^2 = a^2+b^2$

$$\geq \frac{1}{4}\langle\psi|\{\hat{q},\hat{p}\}|\psi\rangle + \frac{\hbar^2}{4}$$

$$(\Delta q)^2 (\Delta p)^2 \geq \frac{\hbar^2}{4}$$

[since first term is always positive or zero]

$$(\Delta q)(\Delta p) \geq \frac{\hbar}{2}$$ ✓ ✓

41.6 Prove that gaussian saturates the inequality i.e for gaussian

$$\Delta x . \Delta p = \frac{\hbar}{2}$$

41.6. Continued :

We used Schwarz Inequality

$$|V_1|^2 \, |V_2|^2 \geq |\langle V_1 | V_2 \rangle|^2$$

This becomes equality only when $|V_1\rangle = c\,|V_2\rangle$ for some complex number c.

$\text{Re}(\langle V_2|V_1\rangle) = \text{Re}(c\,\langle V_2|V_1\rangle)$ gives 'c' purely imaginary...

$c = ia$, a = some real constant.

i.e in our terms;

$$(p - \langle p \rangle)|\psi\rangle = c\,(q - \langle q \rangle)|\psi\rangle \quad — \quad (7)$$

In position basis (7) becomes

$$\left(-i\hbar \frac{d}{dq} - \langle p \rangle\right)|\psi\rangle = c\,(q - \langle q \rangle)|\psi\rangle$$

$$\left(-i\hbar \frac{d}{dq} - \langle p \rangle\right)\psi(q) = c\,(q - \langle q \rangle)\psi(q)$$

'x' or 'q' means position.

$$\frac{d\psi(x)}{\psi(x)} = \frac{i}{\hbar}\left[\langle p \rangle + c\,(x - \langle x \rangle)\right]dx$$

Integrating the above and noting the fact that 'c' is purely imaginary.

$$\frac{d\psi(x)}{\psi(x)} = \frac{i}{\hbar}\left(iax - ia\langle x \rangle + \langle p \rangle\right)dx$$

$$\frac{d\psi(x)}{\psi(x)} = \frac{a}{\hbar}\left(-x + \langle x\rangle + \frac{i\langle p\rangle}{a}\right) dx$$

$$\ln \psi(x) = \frac{a}{\hbar}\left(-\frac{x^2}{2} + \langle x\rangle x + \frac{i\langle p\rangle}{a} x\right) + \text{constant}$$

'Let it be A'

Now, Let $A = \frac{-\langle x\rangle^2 a}{2\hbar} + B$

[redacted] $\psi(x) = A\, e^{-\frac{a(x-\langle x\rangle)^2}{2\hbar}} e^{\frac{i\langle p\rangle x}{\hbar}}$ $\left[A = e^{B}\right]$

$$\psi(x) = A\, e^{-a\frac{(x-\langle x\rangle)^2}{2\hbar}}\, e^{\frac{i\langle p\rangle x}{\hbar}}$$ ✓

$$= A\, e^{-\frac{(x-\langle x\rangle)^2}{2\Delta^2}}\, e^{\frac{i\langle p\rangle x}{\hbar}}$$ ✓.

Gaussian wave packet.

Hence

$\Delta x . \Delta p = \hbar/2$ is satisfied by gaussian wavepacket.. ✓

4. c. $\psi(p)$ for the state $|\psi\rangle$ in (ii)

$$\psi(p) = \langle p|\psi\rangle = \int \langle p|x\rangle\langle x|\psi\rangle\, dx = \int \psi_p^*(x)\psi(x)\, dx$$

41.c Continued :

$$\langle p|\psi\rangle = \int_{-\infty}^{\infty} \frac{e^{\frac{-ipq}{\hbar}}}{\sqrt{2\pi\hbar}} \exp\left(\frac{i\langle p\rangle q}{\hbar} - \frac{(q-\langle q\rangle)^2}{4\Delta q^2}\right) \frac{1}{(2\pi)^{\frac{1}{4}}(\Delta q)^{\frac{1}{2}}}\, dq$$ ✓

Let us simplify notations.

use $\langle p\rangle = p_0$

$\langle q\rangle = q_0$

$$\langle p|\psi\rangle = \frac{1}{(2\pi)^{\frac{1}{4}}}\frac{1}{(\Delta q)^{\frac{1}{2}}}\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipq}{\hbar}} e^{\frac{ip_0 q}{\hbar}} e^{-\frac{(q-q_0)^2}{4\Delta q^2}}\, dq$$

$$= \frac{1}{(2\pi)^{\frac{1}{4}}}\frac{1}{(\Delta q)^{\frac{1}{2}}}\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-q\left(\frac{i}{\hbar}(p-p_0)\right)} e^{-\frac{(q-q_0)^2}{4\Delta q^2}}\, dq$$

Let $q - q_0 = \xi \Rightarrow q = q_0 + \xi$

$$= \underbrace{\frac{1}{(2\pi)^{\frac{1}{4}}}\frac{1}{(\Delta q)^{\frac{1}{2}}}\frac{1}{\sqrt{2\pi\hbar}}}_{A} e^{-q_0\left(\frac{i}{\hbar}(p-p_0)\right)} \int_{-\infty}^{\infty} e^{-\xi\left(\frac{i}{\hbar}(p-p_0)\right)} e^{-\frac{\xi^2}{4\Delta q^2}}\, d\xi \quad -(A)$$

Now we will prove an integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}\, dx = e^{\frac{\beta^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{\beta x} e^{-\frac{\beta^2}{4\alpha}}\, dx$$

$$= e^{\frac{\beta^2}{4\alpha}} \int_{-\infty}^{\infty} e^{\left(\sqrt{\alpha}x + \frac{\beta}{2\sqrt{\alpha}}\right)^2} dx$$

$$= e^{\frac{\beta^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}}$$

Hence

$$\int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x} \, dx = \sqrt{\frac{\pi}{\alpha}} \, e^{\frac{\beta^2}{4\alpha}} \quad \text{— (B)}$$

ok

using (B) in (A)

$$= A \, e^{\frac{-ipq_0}{\hbar}} \int_{-\infty}^{\infty} e^{\xi\left(\frac{i}{\hbar}(p-p_0)\right)} e^{-\frac{\xi^2}{4\Delta q^2}} \, d\xi$$

— using (B)

$$= A \, e^{\frac{-ipq_0}{\hbar}} \sqrt{\frac{\pi^2 \cdot 4(\Delta q)^2}{1}} \, e^{\frac{-1}{\hbar^2}\frac{(p-p_0)^2 \hbar^2 \cdot (\Delta q)^2}{\hbar^2}}$$

$$= A \, e^{\frac{-ipq_0}{\hbar}} \cdot 2\sqrt{\pi (\Delta q)^2} \, e^{-\frac{(p-p_0)^2 \cdot 1}{4\Delta p^2}}$$

$$= \frac{1}{(2\pi)^{\frac{1}{4}}} \frac{1}{(\Delta q)^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi\hbar}} 2\sqrt{\pi} (\Delta q) \, e^{-\frac{ip\langle q\rangle}{\hbar} - \frac{(p-\langle p\rangle)^2}{4\Delta p^2}}$$

we have used $(\Delta q)^2 = \frac{\hbar^2}{(\Delta p)^2}$ Gaussian wavepacket condition

$$= \frac{1}{(2\pi)^{\frac{1}{4}}} \frac{2\sqrt{\pi} \, (\Delta q)^{\frac{1}{2}}}{\sqrt{2\pi\hbar}} \, e^{-\frac{ip\langle q\rangle}{\hbar} - \frac{(p-\langle p\rangle)^2}{4\Delta p^2}}$$

$$= \frac{1}{(2\pi)^{\frac{1}{4}}} \cdot \frac{2\sqrt{\pi}}{\sqrt{2\pi}\sqrt{\hbar}} \sqrt{\Delta q}\; e^{\frac{-ip\langle q\rangle}{\hbar}} e^{-\frac{(p-\langle p\rangle)^2}{4\Delta p^2}}$$

Now again use $\Delta q\, \Delta p = \hbar/2$

i.e $\sqrt{\frac{\Delta q}{\hbar}} = \frac{1}{\sqrt{2}\sqrt{\Delta p}}$

$$= \frac{1}{(2\pi)^{1/4}} \frac{1}{(\Delta p)^{\frac{1}{2}}} \exp\left(\frac{-ip\langle q\rangle}{\hbar} - \frac{(p-\langle p\rangle)^2}{4\Delta p^2}\right)$$

Therefore,

$$\langle p|\psi\rangle = \psi(p) = \frac{1}{(2\pi)^{\frac{1}{4}}} \frac{1}{(\Delta p)^{\frac{1}{2}}} \exp\left(\frac{-ip\langle q\rangle}{\hbar} - \frac{(p-\langle p\rangle)^2}{4\Delta p^2}\right)$$

a gaussian again.

3/4. 42. $\psi(t_0, p) \equiv \langle p | \psi(t_0) \rangle$

Solution :

$$= \frac{1}{(2\pi)^{1/4}} \frac{1}{(\Delta p)^{\frac{1}{2}}} \exp\left(-i \frac{\langle q \rangle p}{\hbar} - \frac{(p - \langle p \rangle)^2}{4\Delta p^2} \right)$$

Let $\langle p \rangle = p_0$ and $\langle q \rangle = q_0$ [**] ok

We take the time evolution of $\psi(t_0, p)$ and then take Fourier transform of the resulting $\psi(t, p)$ to get $\psi(x, t)$

$$\psi(x,t) = \int \frac{dp}{(2\pi\hbar)^{1/2}} \exp\left(\frac{ipx}{\hbar} - \frac{ip^2(t-t_0)}{2m\hbar} \right) \psi(t_0, p) \checkmark$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi)^{1/4}} \frac{1}{(\Delta p)^{\frac{1}{2}}} \int dp \exp\left(\frac{ipx}{\hbar} - \frac{ip^2(t-t_0)}{2m\hbar} \right) \cdot \exp\left(\frac{-iq_0 p}{\hbar} - \frac{(p-p_0)^2}{4\Delta p^2} \right) \checkmark$$

Let us shift the momentum coordinates
$$p - p_0 = p'$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi)^{1/4}} \frac{1}{(\Delta p)^{\frac{1}{2}}} \int dp' \exp\left(\frac{i(p'+p_0)x}{\hbar} - \frac{i(p_0+p')^2(t-t_0)}{2m\hbar} \right) \exp\left(\frac{-iq_0(p_0+p')}{\hbar} - \frac{p'^2}{4\Delta p^2} \right)$$

Now, slight manipulation

$$e^{\frac{i(p'+p_0)x}{\hbar}} e^{-\frac{i(p'+p_0)^2(t-t_0)}{2m\hbar}} = e^{\frac{ip_0x}{\hbar} - \frac{ip_0^2(t)}{2m\hbar}} \times e^{ip'\left(x - \frac{p_0t}{m}\right)} e^{\frac{-ip'^2(t-t_0)}{2m\hbar}}$$

Take for time being $t - t_0 \longrightarrow t$

We'll put it back later.

We use $\Delta x \cdot \Delta p = \frac{\hbar}{2}$ [Gaussian]

Then, we have ✓ why? what is ψ?

$$\psi(x,t) = e^{\frac{ip_0x}{\hbar} - \frac{ip_0^2(t-t_0)}{2m\hbar}} \psi\left(x - \frac{p_0(t-t_0)}{m}\right)$$

Compare to $A e^{i(kx-\omega t)}$

$$\psi(x,t) = e^{\frac{i}{\hbar}\left(p_0x - \frac{p_0^2(t-t_0)}{2m}\right)} \psi\left(x - \frac{p_0(t-t_0)}{m}\right)$$

We see that this has a velocity i.e

why group velocity [$v_g = \frac{p_0}{m}$] (✓) — (4)

$v_g = \frac{d\omega}{dk}$.

Also, rewriting the integral

$$\psi(x,t) = \int_{-\infty}^{\infty} e^{\frac{ip_0}{\hbar}\left(x - \frac{p_0(t-t_0)}{m}\right)} e^{-(p'-p_0)\left(x - x_0 - \frac{p_0t}{m}\right)} e^{-\frac{(p-p_0)^2(\Delta x)^2}{\hbar^2}(1 + \sqcup)}$$ — (5)

(+)ve quantity ↑

Now, we make two important observations.

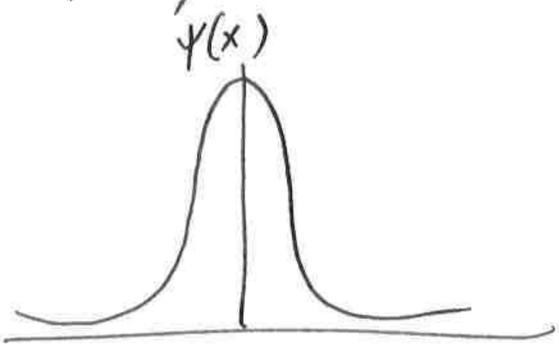
The width of the gaussian wavepacket increases with time since we have

$$\exp\left(\frac{-(p-p_0)^2(\Delta t)^2}{\hbar^2}\right)\left[1 + \underset{F(t)}{\text{positive no.}}\right]$$

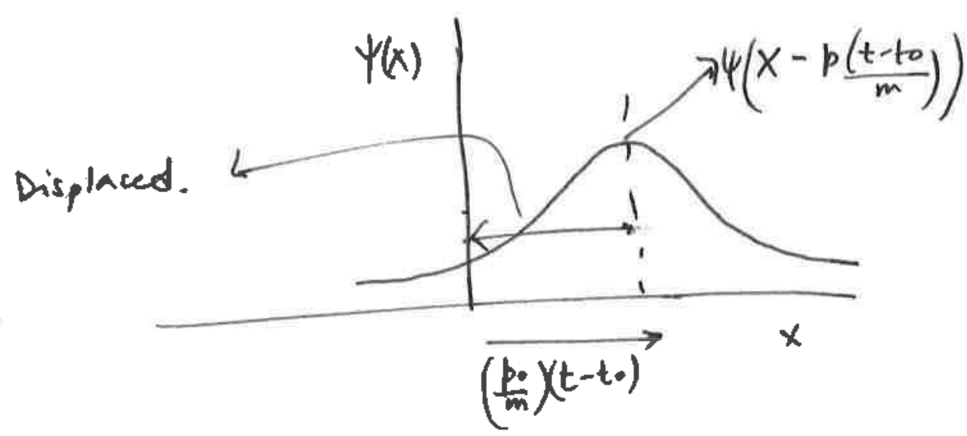
if $t = t_0$, we have original width.

<u>2nd observation</u>: The wavepacket has a group velocity v_g given by $\frac{p_0}{m}$
refer: eq (4)

It is not only a gaussian (Fourier transform of Gaussian is again a Gaussian). But, it moves and spreads.



at $t=0$



42. Continued :-

$\psi(x,t)$ is given by (5) .. ⇐ LAST PAGE.

4/4.

43. Prove the following :

$$U(t,x;t_0,x_o) = \sqrt{\frac{m}{2\pi\hbar i(t-t_0)}}\exp\left(\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right) \quad (5)$$

Solution : $U(t,x;t_0,x_0) = \int \frac{dp}{2\pi\hbar} \exp\left(\frac{i}{\hbar}p(x-x_0)\right)\exp\left(\frac{-i}{\hbar}\frac{p^2}{2m}(t-t_0)\right)$

$$U(t,x;t_0,x_0) = \frac{1}{2\pi\hbar}\int dp\; e^{-p^2\left(\frac{+i}{\hbar}\frac{(t-t_0)}{2m}\right)} e^{p\frac{i}{\hbar}(x-x_0)}$$ ✓

Recall that we proved a standard Gaussian form of integral in Ex: 41.c. We state it here again.

$$\int_{-\infty}^{\infty} dx\; e^{-\alpha x^2 + \beta x} = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$$ ✓

Using the above integral, the expression for $U(t,x;t_0,x_0)$ becomes ;

$$U(t,x;t_0,x_0) = \frac{1}{2\pi\hbar}\sqrt{\frac{\pi \cdot 2m\hbar}{i(t-t_0)}}\; e^{\frac{-(x-x_0)^2}{\hbar^2}\times\frac{2m\hbar}{4i(t-t_0)}}$$ ✓

6

$$= \sqrt{\frac{m}{2\pi\hbar i(t-t_0)}}\exp\left(\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right)$$ ✓.

Proved ...

⇒ TURN OVER

Now we write

$$S_{classical} = \int L_{classical}\, dt \quad ✓$$

$$L_{cl} = \frac{1}{2} m \dot{x}_{cl}^2 \quad ✓ \qquad [\text{free particle}]$$

$$= \frac{1}{2} m \left(\frac{x - x_0}{t - t_0}\right)^2 \quad ✓ \qquad \text{Let } dx = x - x_0, \quad dt = t - t_0$$

$$S \cong L_{av}(t - t_0) \quad ? \Leftarrow \text{For average change in time } (t - t_0)$$

↓ NOT EQUAL EXACTLY

$$S_{classical} \cong \frac{1}{2} m \frac{(x-x_0)^2}{(t-t_0)^2} \times (t - t_0) \quad ✓$$

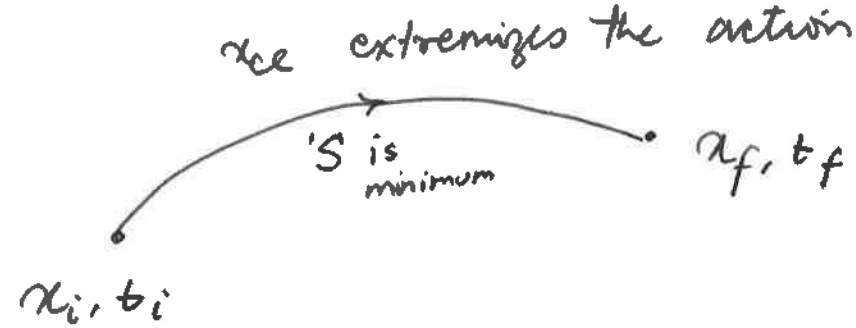
$$\cong \frac{1}{2} m \frac{(x-x_0)^2}{(t-t_0)} \quad ✓$$

$$\frac{i S_{cl}}{\hbar} \cong \frac{i m (x - x_0)^2}{2\hbar (t - t_0)} \quad ✓$$

This is the argument of the exponent in the last page. Hence, we have

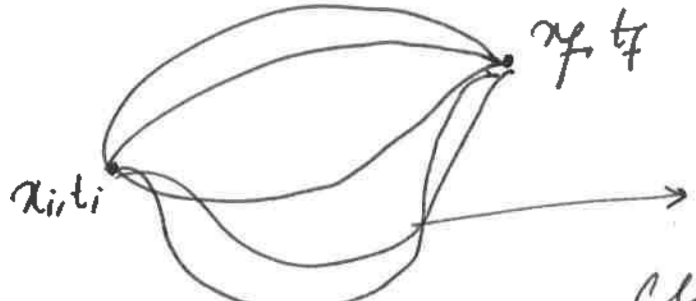
$$U \propto \exp\left(\frac{i S_{cl}}{\hbar}\right) \quad ✓$$

This is the basis of Feynman's path integral formalism.



Classical Mechanics

In quantum, we have contributions to the propagator from other paths too !! (Actually, all the paths that connect x_i, t_i to x_f, t_f.



These paths are not precise like Classical. Gives rise to Heisenberg uncertainity principle

$$\sum_{\text{sum all paths}} e^{\frac{iS}{\hbar}} = U(x_i, t_i; x_f, t_f) \quad ✓$$

$\Downarrow$

Evolution operator

<u>OR</u>

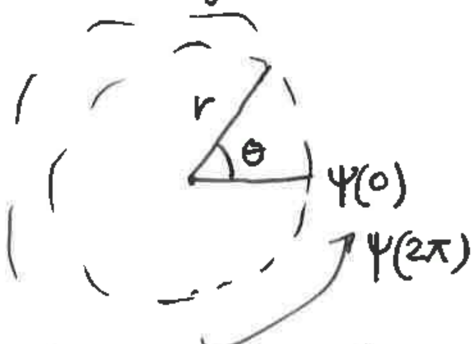
Feynman Propagator..

3/4.

44. Find the energy eigenvectors and energy eigenvalues of a free particle living on a circle of radius R. Is the momentum operator Hermitian ?

Solution :

Schrödinger equation $-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\phi) + 0 = E\psi(r,\theta,\phi)$ ← spherical in polar ✓



Boundary conditions on the wavefunction are that ✓

$$\boxed{\psi(\theta = 0°) = \psi(\theta = 2\pi)}$$

Since, here the ψ only depends on 'θ' explicitly. we solve the Schrödinger eqn in 'θ' coordinates.

In general, $\psi(\phi) = \psi(\phi + 2\pi)$

$\psi(\theta) = \psi(\theta + 2\pi)$ ← in our problem.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \checkmark$$

[We don't consider the φ, r coordinates since 'ψ' is independent of them]

$$-\frac{\hbar^2}{2mr^2}\frac{\partial^2\psi(\theta)}{\partial\theta^2} = E\psi(\theta)$$

$$\frac{\partial^2\psi(\theta)}{\partial\theta^2} = -\frac{2mEr^2}{\hbar^2}\psi(\theta) \Rightarrow \frac{d^2\psi}{d\theta^2} = -m^2\psi$$

where $m^2 = \frac{2mEr^2}{\hbar^2}$

Now $\psi(\phi + 2\pi) = \psi(\phi)$ use n (m is mass)

$\Rightarrow e^{im(\theta+2\pi)} = e^{im\theta} \Rightarrow m = 0, \pm 1, \pm 2 \ldots$

$$\boxed{\psi(\theta) = Ae^{\pm im\theta}} \checkmark$$

↳ Solution to Schrodinger equation.

Now, since $m^2 = \frac{2mEr^2}{\hbar^2}$

$$\Rightarrow E = \frac{m^2 \hbar^2}{2mr^2}$$

Define $mr^2 = I$ (Moment of inertia)

$$\Rightarrow \boxed{E = \frac{m^2 \hbar^2}{2I}} \rightarrow \text{Energy eigenvalue}$$

$$\psi(\theta) = A e^{\pm i \frac{\sqrt{2mE}\, r}{\hbar} \theta}$$

$$\psi(0) = 0 \Rightarrow A = 0$$

$$\boxed{\psi(\theta) = e^{\pm i \frac{\sqrt{2mE}\, r\theta}{\hbar}}} \leftarrow \text{Eigenvector.}$$

b) Is $\hat{p}$ hermitian?

$$\hat{p} = -\frac{i\hbar}{R} \frac{\partial}{\partial \theta}$$

To be hermitian

$$\int (\hat{p}\psi)^* \psi \, dx = \int \psi^* \hat{p} \psi \, dx$$

$$\hat{p} = \frac{-i\hbar}{R}\frac{\partial}{\partial\theta} \quad \text{and} \quad \psi = \psi(\theta)$$

$$\frac{+i\hbar}{R}\int_0^{2\pi}\left(\frac{\partial\psi(\theta)}{\partial\theta}\right)^*\psi(\theta)\,d\theta = \int_0^{2\pi}[\psi(\theta)]^*\frac{-i\hbar}{R}\frac{\partial\psi(\theta)}{\partial\theta}\,d\theta$$

$$\frac{+i\hbar}{R}\,\psi(\theta)\psi(\theta)^*\Big|_0^{2\pi} - \frac{i\hbar}{R}\int\frac{d\psi(\theta)}{d\theta}\psi(\theta)^*\,d\theta$$

$$= \frac{-i\hbar}{R}\psi^*(\theta)\psi(\theta)\Big|_0^{2\pi} - \frac{i\hbar}{R}\int\frac{d\psi(\theta)}{d\theta}\psi(\theta)^*d\theta$$

They are equal, only if

$$\psi(0) = \psi(2\pi) \cancel{= 0}$$

i.e wavefunction vanishes at boundaries?

Hence, considering the boundary conditions that $\psi(0) = \psi(2\pi) = \cancel{0}$; $\hat{p}_\theta$ is Hermitian

p_θ is <u>HERMITIAN</u> ....

22. Expectation of A defined as (33/46) RAGHAV G. JHA

09/26/13

4/6.

$$\langle A \rangle = \sum_i W_i \langle \alpha^{(i)} | A | \alpha^{(i)} \rangle$$ (why?)

$$= \sum_i \sum_{a'} W_i \langle \alpha^{(i)} | a' \rangle \langle a' | A | \alpha^{(i)} \rangle$$

← Put II here

we can write using more general basis b'

$$= \sum_i W_i \sum_{b'} \sum_{b''} \langle \alpha^{(i)} | b' \rangle \langle b' | A | b'' \rangle \langle b'' | \alpha^{(i)} \rangle$$

$$= \sum_{b'} \sum_{b''} \left(\sum_i W_i \langle b'' | \alpha^{(i)} \rangle \langle \alpha^{(i)} | b' \rangle \right) \langle b' | A | b'' \rangle$$

But $\rho = \sum_i W_i | \alpha^{(i)} \rangle \langle \alpha^{(i)} |$, so we have

$$= \sum_{b'} \sum_{b''} \langle b'' | \rho | b' \rangle \langle b' | A | b'' \rangle$$ II

$$= \sum_{b''} \langle b'' | \rho A | b'' \rangle$$

$$= Tr(\rho A)$$ (✓)

21. In the bra-ket formalism, the probability of outcome a is

2/6.

$$p_a = \langle a | P_a | a \rangle$$ (ψ) ?

...... ① where P_a projects onto $|a\rangle$ i.e eigenspace of a

$$= \langle a | P_a | a \rangle$$

$$= \sum_{a'} \langle a | a' \rangle \langle a' | P_a | a \rangle$$ ✓

($\mathbb{I}$ marked under $|a'\rangle\langle a'|$)

$$= \sum_{a'} \langle a' | P_a | a \rangle \langle a | a' \rangle$$

Changing the order of product?

(ρ marked under $|a\rangle\langle a|$)

$$= \sum_{a'} \langle a' | \rho P_a | a' \rangle$$ ✓

$$= Tr(\rho P_a)$$ ✓.

Mixed state?

(23) State of an ensemble immediately after an ideal measurement of observable $\underline{A}$ with outcome $\underline{a}$ is

2/8.

$$\rho' = \frac{1}{tr(\rho P_a)} P_a \rho P_a$$

Proof: As discussed before in the class;

ok.

$$|\psi'\rangle = \frac{P_a |\psi\rangle}{\sqrt{\langle \psi | P_a | \psi \rangle}} \quad — (1)$$

$$\langle\psi'| = \frac{\langle\psi|P_a^+}{\sqrt{\langle\psi|P_a|\psi\rangle}} \quad ✓ \qquad - \quad (2)$$

Using (1) & (2)

we can write $\rho' = |\psi'\rangle\langle\psi'|$ as,

for a pure state!

$$\rho' = \frac{P_a|\psi\rangle\langle\psi|P_a}{\langle\psi|P_a|\psi\rangle} \qquad \text{Since } P_a^{+} = P_a \quad ✓$$

mixed?

But $\langle\psi|P_a|\psi\rangle = Tr(\rho P_a)$ [proved in Ex. 2.1

$$= \frac{P_a|\psi\rangle\langle\psi|P_a}{Tr(\rho P_a)}$$

$$= \frac{P_a \rho P_a}{Tr(\rho P_a)} \qquad \underline{\rho = |\psi\rangle\langle\psi|} \quad ✓.$$

Now, if any outcome; then we need to sum P_a's over all $\underline{a}$'s.

i.e $\rho' = \dfrac{\sum_a P_a |\psi\rangle\langle\psi| P_a}{\sum_a Tr(\rho P_a)}$

$= \dfrac{\sum_a P_a \rho P_a}{tr\left(\sum_a \rho P_a\right)}$ ✓ mixed state?

24. Schrödinger eq^n says $\left[i\hbar \dfrac{d\rho(t)}{dt} = [H, \rho(t)] \right]$ ← To prove

4/4.

$i\hbar \dfrac{\partial}{\partial t} |j\rangle = \hat{H} |j\rangle$ ✓ where $|j\rangle \in \mathcal{H}$ — EQ ①

Taking dual of this gives $-i\hbar \langle j| \dfrac{\overleftarrow{\partial}}{\partial t} = \langle j| \hat{H}^{+}$ ✓ ok — EQ ② ← Back derivative; acts from back on the bra

↳ Using $H^+ = H$

Now $\rho = \sum_i W_i |i\rangle\langle i|$

Taking derivative w.r.t 't'

$i\hbar \dfrac{\partial \rho}{\partial t} = i\hbar \dfrac{\partial}{\partial t} \left\{ \sum_j W_j |j\rangle\langle j| \right\}$

$= i\hbar \sum_j \dfrac{\partial W_j}{\partial t} |j\rangle\langle j|$ ✓

$+ i\hbar \sum_j W_j \left(\dfrac{\partial}{\partial t}|j\rangle\right)\langle j| + i\hbar \sum_j W_j |j\rangle \dfrac{\partial}{\partial t}\langle j|$

So, we have

$$i\hbar \frac{\partial \rho}{\partial t} = i\hbar \sum_j \frac{\partial W_j}{\partial t} |j\rangle\langle j| + i\hbar \sum_j W_j \left(\frac{\partial}{\partial t}|j\rangle\right)\langle j|$$

$$+ i\hbar \sum_j W_j |j\rangle \frac{\partial}{\partial t} \langle j|$$

Now, use EQ ① & EQ ②

and the fact that $\frac{\partial W_j}{\partial t} = 0$; stable equilibrium → zero by def.

we get

$$i\hbar \frac{\partial \rho}{\partial t} = \sum_j W_j H |j\rangle\langle j| - \sum_j W_j |j\rangle\langle j| H$$

$$= \sum_j W_j (H\rho - \rho H)$$ ✓

$$= [H, \rho(t)]$$

Hence

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho(t)]$$ ✓

25. i) $U(t, t_0) = \exp\left[\frac{-i}{\hbar} H \cdot (t - t_0)\right]$

25

$$\frac{dU}{dt} = \frac{-i}{\hbar} H \exp\left[\frac{-i}{\hbar} H \cdot (t - t_0)\right]$$ ← Taking derivative w.r.t t

(why?)

6/6.

$$i\hbar \frac{dU}{dt} = H \exp\left[\frac{-i}{\hbar} H \cdot (t - t_0)\right]$$

$$= H U(t, t_0)$$ ✓

Therefore,

$$i\hbar \frac{dU}{dt} = HU(t, t_0)$$ ✓

ii) $U^{+} = U^{-1} \Rightarrow U$ is unitary

Let's prove that UU^{+} is $\mathbb{1}$

why?

$$U = e^{\frac{-i}{\hbar} H \cdot (t - t_0)}, \quad U^{+} = e^{\frac{i}{\hbar} H \cdot (t - t_0)}$$

$$UU^{+} = e^{\frac{-i}{\hbar} H \cdot (t - t_0)} e^{\frac{i}{\hbar} H \cdot (t - t_0)}$$ ✓

Clearly $[\hat{H}, \hat{H}] = 0$ ✓

$\Big\{ e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B}}$ if $[A, B] = 0$ from BCH formula

Therefore

$$UU^{+} = e^{-\frac{i}{\hbar} H \cdot (t - t_0) + \frac{i}{\hbar} H \cdot (t - t_0)}$$

$$= e^{0}$$ ✓

$$= \mathbb{1}$$ ✓ $\Rightarrow$ $\underline{U^{+} = U^{-1}}$ ✓

iii) $U(t_2, t_0) = U(t_2, t_1)\, U(t_1, t_0)$

t_1 is some time between t_0 & t_2

i.e $t_0 < t_1 < t_2$

$$L.H.S = U(t_2, t_0)$$

$$= \exp\left[\frac{-i}{\hbar} H.(t_2 - t_0)\right]$$

$$= \exp\left[\frac{-i}{\hbar} H.\{(t_2 - t_1) + (t_1 - t_0)\}\right]$$ → writing by breaking at t_1

$$= \left\{\exp\left[\frac{-i}{\hbar} H.(t_2 - t_1)\right] \times \exp\left[\frac{-i}{\hbar} H.(t_1 - t_0)\right]\right\}$$

$[H, H] = 0$

$e^{\hat{A}+\hat{B}} = e^{\hat{A}} \cdot e^{\hat{B}}$

$$= U(t_2, t_1)\, U(t_1, t_0)$$

So, we can consider the evolution from 't_0' to 't_f' by slicing it into 'n' parts i.e $\frac{t_f - t_0}{N}$

Actually, this applies to Feynman propagator / Green's function / kernel since

$$\mathcal{A}(x_f, t_f; x_i, t_i) = \int dx_a\, \mathcal{A}(x_f, t_f; x_a, t_a)\, \mathcal{A}(x_a, t_a; x_i, t_i)$$

26. $U(t,t_0) = \mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1 H(t_1) + \sum_{n=2}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 H(t_1) \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_n)$

(26)

3/14. We start away take the derivative of $U(t,t_0)$ w.r.t t ok

$$\frac{dU}{dt} = \frac{d}{dt}\left(\mathbb{1} - \frac{i}{\hbar}\int_{t_0}^{t} dt_1 H(t_1) + \sum_{n=2}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 H(t_1) \cdot \int_{t_0}^{t_{n-1}} dt_n H(t_n) + \cdots\right.$$

$$= 0 - \frac{i}{\hbar}\frac{d}{dt}\int_{t_0}^{t} dt_1 H(t_1) + \left(\frac{-i}{\hbar}\right)^2 \frac{d}{dt}\int_{t_0}^{t} dt_1 H(t_1) \cdot \int_{t_0}^{t_1} dt_2 H(t_2) + \cdots$$

EQ (1)

Now we do one term at a time

$$\frac{-i}{\hbar}\frac{d}{dt}\int_{t_0}^{t} dt_1 H(t_1) = \frac{-i}{\hbar}\int_{t_0}^{t}\frac{d}{dt} \underline{H(t_1)\, dt_1}$$

$$= \frac{-i}{\hbar}\frac{d}{dt}\left[\mathcal{H}(t) - \mathcal{H}(t_0)\right]$$

Using fundamental theorem of calculus. Integral of H, represent by $\underline{\mathcal{H}}'$ (curly $\mathcal{H}$)

directly.

$$= \frac{-i}{\hbar} H(t) \quad \cdots \text{(A)}$$

Now second term

$$-\left(\frac{i}{\hbar}\right)^2 \frac{d}{dt}\int_{t_0}^{t} dt_1 H(t_1) \int_{t_0}^{t_1} dt_2 H(t_2)$$

← Use product rule

$d(uv) = u\,dv + v\,du$

$$\frac{-1}{\hbar^2}\left[\frac{d}{dt}\int_{t_0}^{t} H(t_1)dt_1 \times \int_{t_0}^{t_1} dt_2 H(t_2) + \int_{t_0}^{t} H(t_1)dt_1 \times \frac{d}{dt}\int_{t_0}^{t_1} dt_2 H(t_2)\right]$$

$$= \frac{-1}{\hbar^2}\left[H(t) \times \int_{t_0}^{\not{t_1}\ t} dt_2 H(t_2) + \int_{t_0} H(t_1)dt \times 0\right]$$

$$= \frac{-1}{\hbar^2}\left[H(t)\int_{t_0}^{\not{t_1}\ t} dt_2 H(t_2) + \dots\right] \quad — \text{(B)}$$

using (A) & (B) in Eq. (1)

other terms?

$$\frac{dU}{dt} = \frac{-i}{\hbar}H(t) - \frac{1}{\hbar^2}H(t)\int_{t_0}^{t} dt_2 H(t_2) \dots$$

$$\frac{dU}{dt} = \frac{-i}{\hbar}H(t)\left[\mathbb{1} - \left(\frac{i}{\hbar}\right)\int_{t_0}^{t} dt_2 H(t_2) \dots\right]$$

$\downarrow$ $U(t, t_0)$

$$i\hbar\frac{dU}{dt} = H(t)\,U(t, t_0)$$

hence proved.

27. (27) Show $i\hbar \frac{d}{dt}\langle \hat{A} \rangle = \langle [\hat{A},\hat{H}] \rangle + i\hbar \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$

4.14.

We first compute ;

$$\frac{d}{dt}\langle \hat{A} \rangle = \frac{d}{dt}\langle \psi | \hat{A} | \psi \rangle$$

$$= \left\langle \frac{\partial \psi}{\partial t} \middle| A \middle| \psi \right\rangle + \left\langle \psi \middle| \frac{\partial \hat{A}}{\partial t} \middle| \psi \right\rangle + \left\langle \psi \middle| \hat{A} \middle| \frac{\partial \psi}{\partial t} \right\rangle \quad — (1)$$

ok.

Schrödinger eqⁿ says

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$$ ✓ & $$\left[-i\hbar \langle \psi | \overleftarrow{\frac{\partial}{\partial t}} = \langle \psi | H \;\; \Updownarrow \;\; -i\hbar \left\langle \frac{\partial \psi}{\partial t} \right| = \langle \psi | H \right]$$ ✓

Using in (1) gives,

$$\frac{d}{dt}\langle \hat{A} \rangle = \frac{-1}{i\hbar}\langle \psi | \hat{H}\hat{A} | \psi \rangle + \left\langle \psi \middle| \frac{\partial \hat{A}}{\partial t} \middle| \psi \right\rangle + \frac{1}{i\hbar}\langle \psi | \hat{A}\hat{H} | \psi \rangle$$

$$= \frac{1}{i\hbar}\langle \psi | \hat{A}\hat{H} | \psi \rangle - \frac{1}{i\hbar}\langle \psi | \hat{H}\hat{A} | \psi \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$$

$$= \frac{i}{\hbar}\langle [\hat{H},\hat{A}] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$$ ✓

$$i\hbar \frac{d}{dt}\langle \hat{A} \rangle = -\langle [\hat{H},\hat{A}] \rangle + i\hbar \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$$

$$= \langle [\hat{A},\hat{H}] \rangle + i\hbar \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$$

Proved. ✓

28. $[A, B] = 0$

474. Let $\hat{A}$ satisfy $\hat{A}|\eta\rangle = a|\eta\rangle$ — (1)

Multiplying both sides by $\hat{B}$

$$\hat{B}[\hat{A}|\eta\rangle] = \hat{B}[a|\eta\rangle]$$

$$= a\hat{B}|\eta\rangle$$

$$= a|x\rangle$$ ✓

Assume **

$\hat{B}|\eta\rangle =$ [redacted] $|x\rangle$

$$\hat{B}[\hat{A}|\eta\rangle] = \hat{A}[\hat{B}|\eta\rangle] \quad [\text{since } \hat{A}\hat{B} = \hat{B}\hat{A}]$$

$$= \hat{A}|x\rangle$$

Hence, we get

$$\hat{A}|x\rangle = a|x\rangle$$

i.e $|x\rangle$ $= B|\eta\rangle$ is also an eigenvector of (A) with eigenvalue 'a' ✓

Now, this is true only if (see ** at top)

$|x\rangle = b|\eta\rangle$ why? non-degenerate case.

Therefore, $|\eta\rangle$ and $|x\rangle$ are eigenvectors of both $\hat{A}$ & $\hat{B}$.

28. Continued

Hence in compact form, we can write (/general form)

$$\hat{A}\,|\eta, x ; 2\rangle = a\,|\eta, x ; 2\rangle$$

(η, x ↓ ; 2 ↓ degeneracy is 2 here)

and $\hat{B}\,|\eta, x ; 2\rangle = b\,|\eta, x ; 2\rangle$

We have used a different notation from the prescribed exercise.

EXERCISE	HERE
$\|a, b ; i\rangle$	$\|\eta, x ; N\rangle$ (N → degeneracy fold.)

29. Suppose that $f(x)$, $x \in \mathbb{C}$ admit a series expansion $f(x) = \sum_{n=0}^{\infty} f_n x^n$ and that $\hat{A}|a\rangle = a|a\rangle$. Show that $f(\hat{A}) = \sum_{n=0}^{\infty} f_n \hat{A}^n$ satisfies

4/4

$$f(\hat{A})|a\rangle = f(a)|a\rangle$$ and hence

$$f(\hat{A}) = \sum_a f(a)\,|a\rangle\langle a|$$

Solution: $f(\hat{A}) = \sum_{n=0}^{\infty} f_n \hat{A}^n$ — ①

satisfies $f(\hat{A})|a\rangle = f(a)|a\rangle$

Let's use $f(\hat{A})$ and act it on a state $|a\rangle$

$$f(\hat{A})|a\rangle = \sum_{n=0}^{\infty} f_n \hat{A}^n |a\rangle \quad - ②$$

Now since $\hat{A}|a\rangle = a|a\rangle$, we have

$\hat{A}^n|a\rangle = a^n|a\rangle$ [we can prove this easily by induction]

$n=1$ is given to be true

Assume $n = m-1$ & then prove $n = m$

Using above in eq. ②, we get

$$f(\hat{A})|a\rangle = \sum_{n=0}^{\infty} f_n a^n |a\rangle \quad - ③$$

But since $f(x) = \sum_{n=0}^{\infty} f_n x^n$ (series expansion GIVEN)

Eq. ③ becomes $f(\hat{A})|a\rangle = f(a)|a\rangle$ ✓

HENCE, IT IS SATISFIED.

Now let's post-multiply by a <u>bra</u> $\langle a|$ and sum over all possible a's on both sides

$$f(\hat{A})\sum_a |a\rangle\langle a| = \sum_a f(a)\, |a\rangle\langle a|$$

[But $\sum_a |a\rangle\langle a| = \mathbb{I}$

$f(\mathbb{I}\hat{A}) = f(\hat{A})$ USING THIS]

$$f(\hat{A}) = \sum_a f(a)\, |a\rangle\langle a|$$ ✓

Proved.

Quantum Assignment - 3

Raghav G. Jha

9/19/2013 ~~28~~ 32 / 38

Exercise 12: The probability of measuring an outcome is

4.4.

$$p_a = \langle \psi | P_a | \psi \rangle$$ ✓

Now, since we want probabilities of any outcome (from the spectrum of its eigenvalues) we need to show:

$$\sum_a p_a = 1$$ ✓

$$\Rightarrow \sum \langle \psi | P_a | \psi \rangle = 1$$ ← TO PROVE

Now $\sum_a p_a = \sum_a \langle \psi | a \rangle \langle a | \psi \rangle$ ✓ ($|a\rangle\langle a| \rightarrow P_a$)

$$= \langle \psi | \left(\sum_a |a\rangle\langle a| \right) | \psi \rangle$$ where $|a\rangle$ corresponds to eigenvalue 'a'.

$$= \langle \psi | \psi \rangle$$ ✓

$$= 1$$ ✓. ✓

The sum of all probabilities is $\underline{\underline{1}}$. ✓

Exercise 13: $|\psi\rangle$ and $e^{i\xi}|\psi\rangle$, $\xi \in \mathbb{R}$ yield

4/4. same probabilities for measurement

$$p_a = \langle\psi| P_a |\psi\rangle \quad \text{for } |\psi\rangle \in \mathcal{H}$$ ✓

Now, for unit norm vector $e^{i\xi}|\psi\rangle$

$$p_a = \langle\psi| e^{-i\xi} P_a e^{i\xi} |\psi\rangle$$ ✓

$$= \langle\psi| P_a |\psi\rangle$$ ✓ . ✓

Therefore, both yield the same probability.

EXERCISE 14: Show that $A = \sum_a a|a\rangle\langle a|$

4/4 Let us take an arbitrary state denoted by $|\alpha\rangle$

$$|\alpha\rangle = \sum_a |a\rangle\langle a|\alpha\rangle$$ ok

Now, act on state $|\alpha\rangle$ with A

$$A|\alpha\rangle = \sum_a A|a\rangle\langle a|\alpha\rangle$$ ✓

$$= \left[\sum_a a|a\rangle\langle a|\right] |\alpha\rangle \checkmark$$

$A|a\rangle = a|a\rangle$

Eigenvalue eq^n of observable A.

i.e

$$A|\alpha\rangle = \left[\sum_a a|a\rangle\langle a|\right] |\alpha\rangle$$

$$\left[A - \sum_a a|a\rangle\langle a|\right] |\alpha\rangle = 0$$

Since $|\alpha\rangle$ is arbitrary as said earlier,

$$\Rightarrow \quad A = \sum_a a|a\rangle\langle a| \checkmark$$

✓

EXERCISE 15: Let us denote the characteristic function of

2/4

A as $\phi_X(A)$.

By definition $\phi_X(A)$ = $E[e^{iAX}]$ { i.e expectation value of e^{iAX} }

$\phi_A(x)$ function of x !

Characteristic function is closely related to the Fourier transform.

$$\phi_X(\hat{A}) = \int e^{iAx} p(x)\, dx \qquad \text{for continuous random variable}$$

(blue annotation: "A" written above A and above x in the exponent/differential)

$$= \overline{\int e^{-iAx} p(x)\, dx}$$

$$= \overline{P(A)}$$

$$\boxed{\phi(A) = \int e^{-iAx} p(x)\, dx}$$

This is nothing but the Fourier transform from $\hat{X}$ space to $\hat{A}$ space.

??

$\downarrow$ $\underline{\hbar = 1}$

Suppose: $\hat{A} = \hat{p}$ (momentum)

Then

$$\phi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{ipx}{\hbar}}\, \underline{\psi(x)}\, dx \quad \Bigg\} \text{ Exact fourier transform } \checkmark$$

So, we are convinced about the result ...

about what? result.

EXERCISE 16 : Show that $\langle\psi'|\psi'\rangle = 1$ and

4/4. $\langle\psi'|P_a|\psi'\rangle = 1$.

PROOF :

We have $|\psi'\rangle = \dfrac{P_a|\psi\rangle}{\sqrt{\langle\psi|P_a|\psi\rangle}}$ ✓ — ①

and $\langle\psi'| = \dfrac{\langle\psi|P_a^\dagger}{\left[\sqrt{\langle\psi|P_a|\psi\rangle}\right]^*}$ — ② ✓ since $[P_a^\dagger = P_a]$ ✓

Taking 'bra' & ket as a scalar product

$$\langle\psi'|\psi'\rangle = \frac{\langle\psi|P_a^2|\psi\rangle}{\langle\psi|P_a|\psi\rangle}$$ ✓

$P_a^2 = P_a$ ✓

$$= \frac{\langle\psi|P_a|\psi\rangle}{\langle\psi|P_a|\psi\rangle}$$

$[P_a^2 = P]$ Proved earlier assignment

$= 1$. ✓

Now $\langle\psi'|P_a|\psi'\rangle = \dfrac{\langle\psi|P_aP_aP_a|\psi\rangle}{\sqrt{\langle\psi|P_a|\psi\rangle}\;\sqrt{\langle\psi|P_a|\psi\rangle}^*}$ ✓

$$= \frac{\langle \psi | P_a | \psi \rangle}{\left[\sqrt{\langle \psi | P_a | \psi \rangle}\right]^2}$$

(with $| \psi \rangle$ crossed out in numerator and denominator)

$$= 1 . \checkmark$$

$$\left\{ \frac{P_a^2 = P_a}{P_a^2 P_a = P_a^2} \right.$$

$$P^2 P_a = P_a$$

↓ Using this identity.

EXERCISE 17: Show that

3/4.

$$\frac{d}{dt} \langle \psi(t) | \psi(t) \rangle = 0$$

PROOF: Schrödinger equation says,

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} \hat{H} |\psi(t)\rangle \quad (\checkmark)$$

$(\hbar \equiv 1)$.

$$\frac{d}{dt} \langle \psi(t) | \psi(t) \rangle = \left[\frac{d}{dt} \langle \psi | \right] |\psi\rangle + \langle \psi | \left[\frac{d}{dt} |\psi\rangle \right]$$

Using regular differentiation rules.

$$= i \langle \psi(t) | \hat{H} | \psi(t) \rangle$$

$$-i \langle \psi(t) | \hat{H} | \psi(t) \rangle$$

Little MESSY.

$$= 0 \quad \checkmark$$

FAIRED OUT NEXT PAGE !!

Since

$$\frac{d}{dt}\langle\psi(t)| \neq i\hat{H}^{+}\langle\psi(t)|$$

and $$\frac{d}{dt}|\psi(t)\rangle = -i\hat{H}|\psi(t)\rangle$$ ✓

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = \left[\frac{d}{dt}\langle\psi(t)|\right]|\psi\rangle + \langle\psi|\left[\frac{d}{dt}|\psi(t)\rangle\right]$$

$$= i\langle\psi(t)|H|\psi(t)\rangle - i\langle\psi(t)|H|\psi(t)\rangle$$ ✓

$$= 0$$

[we used $H^{\dagger} = H$ where?]

EXERCISE 18 : Show that $\hat{\vec{L}} = \hat{\vec{L}}^{+}$ $(\hat{L}_i = \hat{L}_i^{+})$

1/4.

$$\vec{L} = \hat{\vec{x}} \times \hat{\vec{p}}$$

$$\hat{L}_i = (\hat{\vec{x}} \times \hat{\vec{p}})_i$$ ✗

i here is an index : i = 1, 2, or 3 !

$$= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$ ✗ —— (Y)

NOW, USING COMMUTATION RELATIONS OF $\hat{\vec{X}}$ & $\hat{\vec{P}}$.

But $[\hat{y}, \hat{p}_z] = 0 \Rightarrow \hat{y}\hat{p}_z = \hat{p}_z\hat{y}$ ✓ — (A)

and $[\hat{z}, \hat{p}_y] = 0 \Rightarrow \hat{z}\hat{p}_y = \hat{p}_y\hat{z}$ ✓ — (B)

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$$= \hat{p}_z\hat{y} - \hat{p}_y\hat{z} \quad \rightarrow \text{using (A) \& (B)}$$

$$= (\hat{L}_x)^\dagger \quad \text{iv)} \quad \text{others?}$$

EXERCISE 19: i) $\rho^\dagger = \rho$

4/4. We have $\rho = \sum_i W_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$

Now take dagger on both sides

$$\rho^\dagger = \sum_i W_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$$

$$= \rho \quad ✓$$

INTERMEDIATE STEP

[We used $(AB)^\dagger = B^\dagger A^\dagger$] $|\alpha^{(i)}\rangle$ is not an operator...

and $W_i^* = W_i$ ✓

$$\rho^\dagger = \sum_i W_i^* \left[|\alpha^{(i)}\rangle\langle\alpha^{(i)}|\right]^\dagger$$

$$= \sum_i W_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}| \quad ✓$$

$$= \rho$$

ii) $\text{tr}\,\rho = 1$

$$\text{tr}\,\rho = \sum_i \langle i|\rho|i\rangle$$ ✓

$$= \sum_i \sum_j P_j \langle i|\psi^{(j)}\rangle\langle\psi^{(j)}|i\rangle$$ ✓ [Since $\rho = \sum_i P_i |\psi^{(i)}\rangle\langle\psi^{(i)}|$

$\rho = \sum_j P_j |\psi^{(j)}\rangle\langle\psi^{(j)}|$

~~$\sum_j \langle\psi^{(j)}|\psi^{(j)}\rangle$~~

~~$= 1$~~

$$= \sum_i \sum_j P_j \langle\psi^{(j)}|i\rangle\langle i|\psi^{(j)}\rangle$$ ✓

$$= \sum_j P_j \langle\psi^{(j)}|\psi^{(j)}\rangle$$ ✓

$$= \sum_j P_j$$ ✓

$$= 1.$$ ✓

EXERCISE 20: Pure $\Longleftrightarrow$ $\rho^2 = \rho$ (both ways)
iff

6/6.

First we will go from

Pure $\Rightarrow$ $\rho^2 = \rho$

* Pure state is represented by $\rho = |\psi\rangle\langle\psi|$ ✓

$$\rho^2 = \rho \cdot \rho$$

$$= |\psi\rangle\langle\psi| \cdot |\psi\rangle\langle\psi|$$ ✓

$$= |\psi\rangle\langle\psi|\psi\rangle\langle\psi|$$ ✓ $\rightarrow \langle\psi|\psi\rangle = 1$

$$= |\psi\rangle\langle\psi|$$ ✓

$$= \rho$$ ✓

So pure state means $\rho^2 = \rho$ ✓

Now, we have to show that if $\rho^2 = \rho$, then the state is PURE.

Proof: Since ρ is Hermitian, the eigenvalues are real and the corresponding eigenvectors $|\lambda_i\rangle$ can be made orthonormal. good!

$$\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$ ✓ Spectral decomposition of 'ρ'

$$\rho^2 = \sum_i \lambda_i^2 |\lambda_i\rangle\langle\lambda_i|$$ ✓

Therefore, $\lambda_i^2 = \lambda_i \Rightarrow$

$$\lambda_i^2 - \lambda_i = 0$$

$$\lambda_i(\lambda_i - 1) = 0$$

$$\lambda_i = 0 \text{ or } 1 \ \ldots$$ ✓

But $tr\rho = \sum_i \lambda_i = 1$ (we make a transformation where the 'ρ' is entirely diagonal)

$\Downarrow$

$\lambda_i = 1$ for some $i = p$ ✓

and $\lambda_i = 0$ for $i \neq p$ ✓

It means ρ is a pure state $|\lambda_p\rangle\langle\lambda_p|$ ✓

or $|\lambda_0\rangle\langle\lambda_i|$ ✓ → PURE STATE.

<u>Second Method:</u> one correct method is enough...

$$\langle\psi|\rho^2|\psi\rangle = \sum_i \langle\psi|\rho|i\rangle\langle i|\rho|\psi\rangle \delta_{\psi,i}$$? [Inserting orthonormal basis $|i\rangle$]

$$= \langle\psi|\rho|\psi\rangle$$

$$\Rightarrow \quad Tr(\rho^2) = Tr(\rho) = 1.$$

Hence, only one diagonal element is $\underline{1}$; rest are all $\underline{0}$.

Hence, $\rho = |\psi\rangle\langle\psi|$ ✓ <u>Pure state</u>. ✓

30. The energy eigenstates are: 1/6.

$$|E_1\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\left(\alpha|\nu_e\rangle + \beta|\nu_\mu\rangle\right) \quad -① \checkmark$$

$$|E_2\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\left(\beta|\nu_e\rangle - \alpha|\nu_\mu\rangle\right) \quad -②$$

① RAGHAV JHA 10/03/2013

Now let us rewrite in term of $|\nu_e\rangle$ and $|\nu_\mu\rangle$

Multiplying ① by α we get and ② by β, we get

$$\alpha^2|\nu_e\rangle + \alpha\beta|\nu_\mu\rangle = \alpha\sqrt{|\alpha|^2+|\beta|^2}\,|E_1\rangle \quad -③$$

$$\beta^2|\nu_e\rangle - \alpha\beta|\nu_\mu\rangle = \beta\sqrt{|\alpha|^2+|\beta|^2}\,|E_2\rangle \quad -④$$

29/40

Adding ③ & ④ we get

$$|\nu_e\rangle = \frac{\alpha|E_1\rangle}{\sqrt{|\alpha|^2+|\beta|^2}} + \frac{\beta|E_2\rangle}{\sqrt{|\alpha|^2+|\beta|^2}} \quad -⑤$$

ok but not necessary.

Now, multiplying ① by β and ② by α and subtracting we get,

ok

$$|\nu_\mu\rangle = \frac{\beta|E_1\rangle}{\sqrt{|\alpha|^2+|\beta|^2}} - \frac{\alpha|E_2\rangle}{\sqrt{|\alpha|^2+|\beta|^2}} \quad -⑥$$

Now doing /operating evolution operator on ① we get,

$$|E_1(t)\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\left(\alpha e^{-\frac{i\hat{H}_1 t}{\hbar}}|\nu_e\rangle + \beta e^{-\frac{i\hat{H}_2 t}{\hbar}}|\nu_\mu\rangle\right) \quad ?$$

Substituting from ⑤ & ⑥ in above we get

??

$$|E_1(t)\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\left(\frac{\alpha^2 e^{-\frac{i\hat{H}_1 t}{\hbar}}}{\sqrt{|\alpha|^2+|\beta|^2}} + \frac{\beta^2 e^{-\frac{i\hat{H}_2 t}{\hbar}}}{\sqrt{|\alpha|^2+|\beta|^2}}\right)|E_1\rangle + \left(\frac{\alpha\beta e^{-\frac{i\hat{H}_1 t}{\hbar}}}{\sqrt{\alpha^2+\beta^2}} - \frac{\alpha\beta e^{-\frac{i\hat{H}_2 t}{\hbar}}}{(\alpha^2+\beta^2)}\right)|E_2\rangle$$

Now let us calculate the probability of neutrino changing flavour i.e

$$P_{\nu_e \to \nu_\mu};t = \left|\langle E_2 | E_1(t)\rangle\right|^2 \;??$$

$$= \frac{\alpha^2\beta^2}{\sqrt{|\alpha|^2+|\beta|^2}}\left(e^{-\frac{iE_1 t}{\hbar}} - e^{-\frac{iE_2 t}{\hbar}}\right)\left(e^{\frac{iE_1 t}{\hbar}} - e^{\frac{iE_2 t}{\hbar}}\right)$$

$$= \frac{|\alpha|^2|\beta|^2}{\sqrt{|\alpha|^2+|\beta|^2}}\left[1 - e^{\frac{i(E_2-E_1)t}{\hbar}} - e^{\frac{i}{\hbar}(E_1-E_2)t} + 1\right]$$

$$= \frac{|\alpha|^2|\beta|^2}{\sqrt{|\alpha|^2+|\beta|^2}}\left[2 - 2\left(\frac{e^{\frac{i(E_2-E_1)t}{\hbar}} - e^{\frac{i}{\hbar}(E_1-E_2)t}}{2}\right)\right]$$

$$= \frac{2|\alpha|^2|\beta|^2}{\sqrt{|\alpha|^2+|\beta|^2}}\left[1 - \cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right] \;?$$

$$= \frac{2|\alpha|^2|\beta|^2}{\sqrt{|\alpha|^2+|\beta|^2}}\left[1 - \left\{2\cos^2\frac{(E_2-E_1)t}{\hbar} - 1\right\}\right]$$

$$= \frac{4|\alpha|^2|\beta|^2}{\sqrt{|\alpha|^2+|\beta|^2}}\left[\sin^2\left(\frac{(E_2-E_1)t}{\hbar}\right)\right] \quad ** \quad - (7)$$

Now, we could have used the mixing matrix as

$$U = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}; \quad \text{where } |\alpha|^2+|\beta|^2 = 1$$

Since $\cos^2\theta + \sin^2\theta = 1$ ✓

Then eq. (P) will reduce to

$$P_{\nu_e \to \nu_\mu ; t} = 4 \sin^2\theta \cos^2\theta \left[\sin^2 \frac{(E_2 - E_1)t}{\hbar} \right]$$

$$= (\sin 2\theta)^2 \sin^2 \left(\frac{(E_2 - E_1)t}{\hbar} \right)$$

which is the actual oscillation result (standard result).

Now, since we are asked to detect $|\nu_e\rangle$ at later time 't'

The probability required is $\underline{P_{\nu_e \to \nu_e ; t}}$

which is

$$P_{\nu_e \to \nu_e ; t} = 1 - P_{\nu_e \to \nu_\mu ; t} \quad \text{ok.}$$

$$= 1 - 4 \frac{|\alpha|^2 |\beta|^2}{|\alpha|^2 + |\beta|^2} \left[\sin^2 \left(\frac{(E_2 - E_1)t}{\hbar} \right) \right]$$

precisely $(|\alpha|^2 + |\beta|^2)^2$

assuming normalized!!

— from (7) ✓

we see that if $t = 0$

$$P_{\nu_e \to \nu_e} = 1$$

with time it decreases & then again increases.

$\Rightarrow$ OSCILLATE.

31. Show $i\hbar \frac{d\hat{A}_H}{dt} = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar\left(\frac{\partial \hat{A}_S}{\partial t}\right)_H$

4/6

We have $\hat{A}_H = U^\dagger(t,t_0)\hat{A}_S U(t,t_0)$ ✓

$$\frac{d\hat{A}_H}{dt} = \frac{d}{dt}\left(U^\dagger(t,t_0)A_S U(t,t_0)\right)$$

not in general.

$$\neq \frac{i}{\hbar} e^{\frac{i\hat{H}.(t-t_0)}{\hbar}} \hat{H} A_S e^{-\frac{i\hat{H}(t-t_0)}{\hbar}} - \frac{i}{\hbar} e^{+\frac{i\hat{H}.(t-t_0)}{\hbar}} A_S \hat{H} e^{-\frac{i\hat{H}.(t-t_0)}{\hbar}} + e^{\frac{i\hat{H}.(t-t_0)}{\hbar}} \frac{\partial A_S}{\partial t} e^{-\frac{i\hat{H}.(t-t_0)}{\hbar}}$$

how? normal bracket

$$= \frac{i}{\hbar}\left[\hat{H}_H(t) A_H(t) - A_H(t)\hat{H}_H(t)\right] + \left(\frac{\partial A_S}{\partial t}\right)_H$$

$$= \frac{i}{\hbar}\left[\hat{H}_H(t), \hat{A}_H(t)\right] \rightarrow \text{commutator} + \left(\frac{\partial A_S}{\partial t}\right)_H$$

Multiply through out by $i\hbar$

$$i\hbar \frac{d\hat{A}_H}{dt} = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar\left(\frac{\partial A_S}{\partial t}\right)_H$$

✓

Proved.

32. Given $[H, B] = -i\omega A$

$[H, A] = i\omega B$

4/4

We know that

$$\langle \hat{A}(t_0)\rangle = \langle \psi(t_0)|\hat{A}|\psi(t_0)\rangle \checkmark$$

$$\langle A(t)\rangle = \langle \psi(t_0)|U^{+}(t,t_0)\hat{A}\,U(t,t_0)|\psi(t_0)\rangle$$

[Using evolution of the state over time]

Now, we recall from previous exercises that BCH formula reads

$$e^{X} Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \cdots \quad -①$$

Rewriting $\langle A(t)\rangle$ neatly & explicitly..

$$\langle A(t)\rangle = \langle \psi(t_0)|e^{+\frac{i}{\hbar}\hat{H}.(t-t_0)} A\, e^{\frac{-i}{\hbar}\hat{H}.(t-t_0)}|\psi(t_0)\rangle \quad - ②$$

Apply BCH formula

$$e^{\frac{i}{\hbar}\hat{H}.(t-t_0)} A\, e^{\frac{-i}{\hbar}\hat{H}.(t-t_0)} = A + \left[\frac{i}{\hbar}\hat{H}.(t-t_0), A\right] + \frac{1}{2}\left[\frac{i}{\hbar}\hat{H}.(t-t_0), \left[\frac{i}{\hbar}\hat{H}.(t-t_0), A\right]\right] + \cdots$$

$$= A - \frac{i}{\hbar}\left[\hat{H}.(t-t_0), A\right] + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2\left[H.(t-t_0), [H(t-t_0), A]\right]$$

$$= A - \frac{i}{\hbar}\hat{B}.(t-t_0).i\omega - \frac{1}{2\hbar^2}(t-t_0)^2[H, i\omega B] + \cdots$$

$$= A - \frac{i}{\hbar}\hat{B}.(t-t_0).i\omega - \frac{1}{2\hbar^2}(t-t_0)^2 . i\omega(-i\omega A)$$

$$= A + \frac{\omega}{\hbar}\hat{B}.(t-t_0) - \frac{1}{2\hbar^2}(t-t_0)^2.\omega^2 A + \cdots$$

[using $[H,A] = i\omega B$, $[H,B] = -i\omega A$]

Plug this back to ②

$$\langle A(t)\rangle = \langle \psi(t_0) | A + \frac{\omega}{\hbar}\hat{B}.(t-t_0) - \frac{1}{2\hbar^2}\omega^2 \hat{A}.(t-t_0)^2 | \psi(t_0)\rangle$$

$$= \langle A(t_0)\rangle + \frac{\omega}{\hbar}\langle \psi(t_0) | \hat{B}.(t-t_0) | \psi(t_0)\rangle \cdots - \frac{\omega^2}{2\hbar^2}\langle \psi(t_0) | \hat{A}.(t-t_0)^2 | \psi(t_0)\rangle$$

Similarly for $\langle B(t)\rangle$ we get,

$$\langle B(t)\rangle = \langle B(t_0)\rangle - \frac{\omega}{\hbar}\langle \psi(t_0) | \hat{A}.(t-t_0) | \psi(t_0)\rangle + \frac{\omega^2}{2\hbar^2}\langle \psi(t_0) | \hat{B}.(t-t_0)^2 | \psi(t_0)\rangle$$

There is a definite symmetry involved here

$A \rightleftarrows -B$; we find this from $[H,A] = i\omega B$ ✓
$[H,B] = -i\omega A$ ✓

Second method could be using

$$i\hbar \frac{d\langle A\rangle}{dt} = \langle [H, A] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$$

directly...

It gives the same expression for $\langle A(t)\rangle$ and $\langle B(t)\rangle$ as found earlier.

——— x ——— x ———

This is not the whole picture ...

Now, when we did the BCH formula we dropped terms after 2^{nd} term. Interesting thing lies hidden afterwards.

We re-do it not holding ourselves to the second term..

P.T.O

$$\langle A(t)\rangle = \langle \psi(t_0)| U^\dagger(t,t_0)\, A\, U(t,t_0) |\psi(t_0)\rangle$$

Now use BCH formula to $U^\dagger(t,t_0) A U(t,t_0)$

We know

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[\ldots]$$

$$U(t,t_0) = e^{\frac{i}{\hbar}\hat{H}.(t-t_0)}$$

$$e^{\frac{i}{\hbar}\hat{H}.(t-t_0)} A e^{-\frac{i}{\hbar}\hat{H}.(t-t_0)} = A + \left[\frac{i}{\hbar}\hat{H}.(t-t_0), A\right] + \frac{1}{2!}\left[\frac{i}{\hbar}\hat{H}.(t-t_0), \left[\frac{i}{\hbar}\hat{H}(t-t_0), A\right]\right]$$

$$= A + \frac{i}{\hbar}(t-t_0)[H,A] + \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2\left[\hat{H}.(t-t_0), [\hat{H}(t-t_0), A]\right] + 3^{rd}\text{ term}$$

$$= A + \frac{i}{\hbar}(t-t_0)\, i\omega B + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 (t-t_0)^2\left[\hat{H}, [\hat{H},\hat{A}]\right] + 3^{rd}\text{ term}$$

set $\frac{i}{\hbar}(t-t_0) = \xi$

$$= A + i\omega\xi B + \frac{1}{2}\xi^2[H, i\omega B] + 3^{rd}\text{ term}$$

$$= A + i\omega\xi B + \frac{i\omega\xi^2}{2}[H, B] + \underline{\quad\quad}$$ → we'll soon figure out

$$= A + i\omega\xi B - \frac{i^2\omega^2\xi^2}{2} A + \underline{\quad\quad}$$

$$= A + i\omega\xi B + \frac{\omega^2\xi^2 A}{2!} + \frac{i^3\omega^3\xi^3 B}{3!} + \cdots$$

Grouping together, we see that

$$= A\left(1 + \frac{\omega^2\xi^2}{2} + \cdots\right) + B\left(i\omega\xi - \frac{i\omega^3\xi^3}{3!} + \cdots\right)$$

Now $\xi = \frac{i}{\hbar}(t - t_0)$

$$= A\left(1 - \frac{\omega^2}{\hbar^2}\frac{(t-t_0)^2}{2!} + \cdots\right) + B\left(\frac{\omega(t-t_0)}{\hbar} - \frac{\omega^3(t-t_0)^3}{\hbar^3\, 3!} + \cdots\right)$$

cosine series expansion

$$= A\cos\left(\frac{\omega(t-t_0)}{\hbar}\right) + B\sin\left(\frac{\omega(t-t_0)}{\hbar}\right)$$

Plugging it back we see that

$$\langle A(t)\rangle = \langle A(t_0)\rangle \cos\frac{\omega(t-t_0)}{\hbar} + \langle B(t_0)\rangle \sin\omega\frac{(t-t_0)}{\hbar}$$ ✓

Hence

$$\langle A(t)\rangle = \langle A(t_0)\rangle \cos\frac{\omega(t-t_0)}{\hbar} + \langle B(t_0)\rangle \sin\frac{\omega(t-t_0)}{\hbar}$$

Clearly at $t = t_0$

we have $\langle A(t)\rangle = \langle A(t_0)\rangle$

✓

Similarly for $\langle B(t)\rangle$ we get

$$\langle B(t)\rangle = \langle B(t_0)\rangle \cos\frac{\omega(t-t_0)}{\hbar} - \langle A(t_0)\rangle \frac{\sin\omega(t-t_0)}{\hbar}$$

Interesting solution
(though a bit complicated)

✓

POSSIBLE
SECOND ^ METHOD.

$$i\hbar\frac{d\langle A\rangle}{dt} = \langle[H, A]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$$

$$i\hbar\frac{d\langle A\rangle}{dt} = i\omega\langle B\rangle \checkmark$$

Taking derivative w.r.t time

$$i\hbar\frac{d^2\langle A\rangle}{dt^2} = i\omega\frac{d}{dt}\langle B\rangle \checkmark \quad -(1)$$

Now, for B

$$i\hbar \frac{d\langle B\rangle}{dt} = [H, B] + \left\langle \frac{\partial \hat{B}}{\partial t} \right\rangle^{\to 0} \checkmark$$

$$i\hbar \frac{d\langle B\rangle}{dt} = -i\omega \langle A\rangle \quad - \quad (2)$$

Take derivative

$$i\hbar \frac{d^2\langle B\rangle}{dt^2} = -i\omega \frac{d}{dt}\langle A\rangle \checkmark \quad - \quad (3)$$

Use (2) in (1)

$$i\hbar \frac{d^2\langle A\rangle}{dt^2} = i\omega \frac{-i\omega}{i\hbar}\langle A\rangle$$

$$i\hbar \frac{d^2\langle A\rangle}{dt^2} = \frac{-i\omega^2}{\hbar}\langle A\rangle$$

$$\frac{d^2\langle A\rangle}{dt^2} = \frac{-\omega^2}{\hbar^2}\langle A\rangle \checkmark$$

Solution should be

$$\langle A(t)\rangle = \langle A(t_0)\rangle \cos \frac{\omega(t-t_0)}{\hbar} + \langle B(t_0)\rangle \sin \frac{\omega(t-t_0)}{\hbar} \checkmark$$

P. T. O

Both the solution yield identical answers. ✓

33. i) Show that $\rho(t) \equiv U(t,t_0)\rho(t_0)U^{\dagger}(t,t_0)$ satisfies the Schrödinger equation

4/6.

$$i\hbar \frac{d\rho(t)}{dt} = [H_S, \rho(t)]$$

PROOF:

$$i\hbar \frac{d}{dt}\left[U(t,t_0)\rho(t_0)U^{\dagger}(t,t_0)\right] ✓$$

not always. ✗

$$i\hbar \cdot \frac{-i}{\hbar}\hat{H}e^{-\frac{i}{\hbar}\hat{H}.(t-t_0)}\rho(t_0)e^{\frac{i}{\hbar}\hat{H}.(t-t_0)} + i\hbar e^{-\frac{i}{\hbar}\hat{H}.(t-t_0)}\rho(t_0)\frac{i}{\hbar}\hat{H}e^{\frac{i}{\hbar}\hat{H}.(t-t_0)}$$

$$= \hat{H}_S\rho(t) + i^2\rho(t)\hat{H}_S$$

$$= \hat{H}_S\rho(t) - \rho(t)\hat{H}_S$$

$$= [H_S, \rho(t)]$$ (✓) ✓

[using the fact that $\hat{H}$ commutes with $e^{\frac{i}{\hbar}\hat{H}.(t-t_0)}$ to switch left, right]

ii) Show that under such unitary evolution, pure states will remain pure

SOLUTION: $\rho(t) = U(t,t_0)\rho(t_0)U^{\dagger}(t,t_0)$

33. ii) We will prove that if we start with a pure state (7)

$\rho(t_0) = |\psi(t_0)\rangle\langle\psi(t_0)|$ i.e $tr(\rho)=1$ and $tr(\rho^2(t_0))=1$?

we retain that property. $\rho^2 \sim \rho$.

$$\rho(t) = U(t,t_0)\rho(t_0)U^+(t,t_0) \checkmark$$

$$tr\,\rho(t) = tr\left[U(t,t_0)\rho(t_0)U^+(t,t_0)\right]$$

$$= tr\left[\rho(t_0)U(t,t_0)U^+(t,t_0)\right]$$

} cyclic property of trace

$$= tr\left[\rho(t_0)\cdot \mathbb{1}\right]$$

$UU^+ = U^+U = \mathbb{1}$

$$= tr\left[\rho(t_0)\right]$$

$$= 1 \checkmark$$

Now

$$tr\,\rho^2(t) = tr\left[\rho(t)\cdot\rho(t)\right]$$

$$= tr\left[U(t,t_0)\rho(t_0)\underbrace{U^+(t,t_0)U(t,t_0)}_{\mathbb{1}}\rho(t_0)U^+(t,t_0)\right]$$

$$= tr\left[U(t,t_0)\rho(t_0)\,\mathbb{1}\,\rho(t_0)U^+(t,t_0)\right]$$

$$= tr\left[U(t,t_0)\rho^2(t_0)U^+(t,t_0)\right]$$

use cyclic property of trace

$$= tr\left[\rho^2(t_0)\right] = 1$$

$\checkmark$ (v)

Hence $tr(\rho(t)) = 1$ and $tr(\rho^2(t)) = 1$ $\Rightarrow$ PURE STATE !!

34. a. $[\hat{q}^i, (\hat{p}_j)^n] = i\hbar n \hat{p}_j^{n-1} \delta^i_j$ — (1)

6/6.

Consider

$$[(\hat{p}_j)^n, \hat{q}^i] = -i\hbar n \hat{p}_j^{n-1} \delta^i{}_j \quad \text{— (A)}$$

The case $\underline{n=0}$ is trivial

for $n=1$, $[\hat{p}_j, q^i] = -i\hbar \delta^i_j$ is true since

this is the standard commutation relation ✓.

Let us consider that this is true for $n-1$ i.e

$$[(\hat{p}_j)^{n-1}, q^i] = -i\hbar (n-1) \hat{p}_j^{n-2} \delta^i_j \quad \text{— (2)}$$

$$= -i\hbar (n-1) \hat{p}_j^{n-2} [p, q]/-i\hbar \quad \text{— (2A)}$$

We will now prove that it is true for $\underline{n}$ on

$$[(\hat{p}_j)^n, q^i] = (\hat{p}_j)^n (q^i) - (q^i)(\hat{p}_j)^n$$

use example 35. $= \hat{p}_j^n q_i - \underline{\hat{p}_j^{n-1} q^i p_j} + \underline{\hat{p}_j^{n-1} q^i p_j} - q^i \hat{p}_j^n$ ✓

[Adding and subtracting underlined terms]

$$= \hat{p}_j^{n-1} [\hat{p}_j q^i - q^i p_j] + [\hat{p}_j^{n-1}, q^i] \hat{p}_j$$

$$= \hat{p}_j^{n-1} [\hat{p}_j q^i - q^i p_j] - i\hbar (n-1) \hat{p}_j^{n-2} \frac{[p, q]}{-i\hbar} \hat{p}_j$$

Now use (2A) i.e ✓

use

$$[(\hat{p}_j)^{n-1}, q^i] = (n-1)\hat{p}_j^{\,n-2}[p,q]$$ as[illegible]

in (3)

$$= \hat{p}_j^{\,n-1}[\hat{p}_j, q^i] + (n-1)\hat{p}_j^{\,n-2}[p,q]\hat{p}_j$$

$$= \hat{p}_j^{\,n-1}[\hat{p}_j, q^i] + (n-1)\hat{p}_j^{\,n-1}[p,q]$$

$$= \{\hat{p}_j^{\,n-1} + n\hat{p}_j^{\,n-1} - \hat{p}_j^{\,n-1}\}[\hat{p}_j, q^i]$$

$$= n\hat{p}_j^{\,n-1}[\hat{p}_j, q^i]$$

$$= -i\hbar n \hat{p}_j^{\,n-1}\delta^i_j$$ ✓ Proved ✓

34. b. Show $[\hat{p}_j, (\hat{q}_i)^n] = -i\hbar n(\hat{q}^i)^{n-1}$

Consider $[(\hat{q}_i)^n, \hat{p}_j] = i\hbar n(\hat{q}^i)^{n-1}$

Case for $n=0$ & $n=1$ are trivial

Let us suppose that this holds for $\underline{n-1}$ i.e

$$[(\hat{q}_i)^{n-1}, \hat{p}_j] = i\hbar(n-1)(\hat{q}^i)^{n-2}\delta^i_j$$

$$= (n-1)(\hat{q}_i)^{n-2}[\hat{q}_i, p^j]$$

We now prove that it is true for n

$$[\hat{q}_i^n, p^j] = \hat{q}_i^n p^j - p^j (\hat{q}_i)^n$$

$$= \hat{q}_i^n p^j - \underline{\hat{q}_i^{n-1} p^j q_i} + \underline{\hat{q}_i^{n-1} p^j q_i} - p^j (\hat{q}_i)^n$$ → Adding & Subtracting

$$= \hat{q}_i^{n-1} [\hat{q}_i, p^j] + [\hat{q}_i^{n-1}, p^j] q_i$$

$$= \hat{q}_i^{n-1} [q_i, p^j] + (n-1) q_i^{n-2} [q_i, p^j] q_i$$

$$= \hat{q}_i^{n-1} [q_i, p^j] + (n-1) q_i^{n-1} [q_i, p^j]$$

$$= n q_i^{n-1} [q_i, p^j]$$

$$= i\hbar n q_i^{n-1} \delta_i^j$$

So, we proved that

$$[p_j, (q^i)^n] = -i\hbar n (q^i)^{n-1}$$ ok. ✓

35. i) $[A, BC] = B[A, C] + [A, B]C$

4/4 Proof:- $[A, BC] = ABC - BCA$

$$= ABC - BAC + BAC - BCA$$ [Adding & subtracting BAC]

$$= (AB - BA)C + B(AC - CA)$$ ✓

$$= [A, B]C + B[A, C]$$ Proved

35. ii) $[A, B+\lambda C] = [A,B] + \lambda[A,C]$

$$[A, B+\lambda C] = AB + \lambda AC - BA - \lambda CA$$
$$= AB - BA + \lambda AC - \lambda CA \quad ✓$$
$$= [A,B] + \lambda[A,C] \quad ✓.$$

36. Show that $[\hat{q}_H^{\,i}, \hat{p}_{Hj}] = i\hbar \delta^i{}_j$

4/4. By definition,

$$\hat{A}_H = U^\dagger(t,t_0)\,\hat{A}_S\,U(t,t_0) \quad ✓$$

We have,

$$[\hat{q}_H^{\,i}, \hat{p}_{Hj}] = \hat{q}_H^{\,i}\hat{p}_{Hj} - \hat{p}_{Hj}\hat{q}_H^{\,i}$$
$$= U^\dagger(t,t_0)\hat{q}^i U(t,t_0) U^\dagger(t,t_0)\hat{p}_j U(t,t_0) \quad ✓$$
$$- U^\dagger(t,t_0)\hat{p}_j U(t,t_0) U^\dagger(t,t_0)\hat{q}^i U(t,t_0)$$

$$= U^\dagger(t,t_0)\hat{q}^i \hat{p}_j U(t,t_0) \quad \leftarrow [\text{Since } UU^\dagger = \mathbb{1}]$$
$$- U^\dagger(t,t_0)\hat{p}_j \hat{q}^i U(t,t_0) \quad ✓$$

$$= \hat{q}^i\hat{p}_j - \hat{p}_j\hat{q}^i \quad ✓$$
$$= i\hbar\delta^i{}_j \quad ✓ .$$

✓ ⟸ q & p commutes with $U(t,t_0)$ & $U^\dagger(t,t_0)$

$$\hat{A}(\hat{q}^i, \hat{p}_j)_H = \hat{A}(\hat{q}^i_H, \hat{p}_{jH})$$

We write

$$\hat{A} = \sum_{\substack{n=n_1 \\ m=m_1}}^{\substack{m_m \\ n_n}} \overset{?}{C}_{n_1 n_2 n_3 \dots n_n m_1 m_2 \dots m_m} \, q^n p^m \Longleftarrow \text{SUM OF PRODUCTS}$$

$$\hat{A}(\hat{q}^i, \hat{p}_j)_H = \checkmark \sum C_{\dots} U^+(t,t_0) q^n p^m U \quad [\text{Writing one term}]$$

$$= C \sum \underbrace{U^+(t,t_0) q^n U(t,t_0)}_{q^i_H} \underbrace{U^+(t,t_0) p^n U}_{\hat{p}_{jH}}$$

Inserting $\mathbb{I}$ for each term

$$= \sum C_{\dots} (q^i_H)^n (\hat{p}_j)^m_H \quad \text{OK}$$

$$= \hat{A}(\hat{q}^i_H, \hat{p}_{jH}) \checkmark$$

37. Show that

i) $U_q \hat{q}^i U_q^{-1} = \hat{q}^i - a^i$

ii) $U_p \hat{p}_j U_p^{-1} = \hat{p}_j - b_j$

SOLUTION: i) we have $U_q = \exp\left(\frac{-i}{\hbar} \vec{\hat{p}} \cdot \vec{a}\right)$

Since, this U_q is hermitian, we have

2/4. $U_q^{-1} = U_q^{\dagger} = \exp\left(\frac{i}{\hbar} \vec{\hat{p}} \cdot \vec{a}\right)$ ✓

$$U_q \hat{q}^i U_q^{-1} = e^{\frac{-i}{\hbar} \vec{\hat{p}} \cdot \vec{a}} \hat{q}^i e^{\frac{i}{\hbar} \vec{\hat{p}} \cdot \vec{a}} \checkmark$$

$$= \hat{q}^i + \left[\frac{-i}{\hbar} \hat{p}_j \vec{a}, q^i\right] + \ldots\ldots$$

$\left\{ e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] \right.$

why do you ignore these?

$$= q^i - \frac{i}{\hbar}\left[\vec{\hat{p}}_j \cdot \vec{a}, q^i\right]$$ → commutator parentheses

$$= q^i - \frac{i}{\hbar}\left[-i\hbar a^j \delta^i_j\right]$$ OK

$$= q^i - \frac{i}{\hbar}\left[-i\hbar a^i\right]$$ → Normal parenthesis

[only non zero (non trivial) when $i = j$

$$= q^i - a^i$$ ✓..

ii) $U_p \hat{p}_j U_p^{-1} = \hat{p}_j - b_j$ ✓

We know that $U_p = \exp\left(\frac{i}{\hbar} \vec{\hat{q}} \cdot \vec{b}\right)$

We have

$$U_p^{-1} = \exp\left(-\frac{i}{\hbar}\,\hat{\vec{q}}\cdot\vec{b}\right)$$

$$U_p\,\hat{p}_j\,U_p^{-1} = e^{\frac{i}{\hbar}\hat{\vec{q}}^i\cdot\vec{b}}\;\hat{p}_j\;e^{-\frac{i}{\hbar}\hat{\vec{q}}^i\cdot\vec{b}} \qquad \text{(eq-1)}$$

Again BCH formula to R.H.S of eq. (1)

$$= \hat{p}_j + \left[\frac{i}{\hbar}\hat{\vec{q}}^i\cdot\vec{b},\ \hat{p}_j\right] \cdots$$

$$= \hat{p}_j + \frac{i}{\hbar}\left[\hat{\vec{q}}^i\cdot\vec{b},\ \hat{p}_j\right]$$

$$= \hat{p}_j + \frac{i}{\hbar}\left\{ i\hbar\, b^i\, \delta^i{}_j \right\}$$

$$= \hat{p}_j - \frac{1}{\hbar}\cdot\hbar\,(b_j)$$

$$= \hat{p}_j - b_j \quad ✓$$

45. $P \hat{p} P^{-1} = -\hat{p}$ 29/32

and $P\, V(\hat{x})\, P^{-1} = V(-\hat{x})$ and hence $P \hat{H} P^{-1} = \hat{H}$

RAGHAV G. JHA
17 Oct 2013

Solⁿ: Lets do $P|p\rangle$ first

4/6. $P|p\rangle = \int dx\, |x\rangle \langle x|P|p\rangle$ ✓ Since $\int dx\, |x\rangle\langle x| = \mathbb{I}$

$= \int dx\, |x\rangle \langle -x|p\rangle$ ✓

$= \int dx\, |x\rangle\, e^{\frac{-ipx}{\hbar}}$ $\langle x|p\rangle = e^{\frac{i}{\hbar}\bar{p}\cdot\bar{x}}$

$= \int dx\, |x\rangle\, e^{\frac{i(-p)x}{\hbar}}$

$= \int dx\, |x\rangle \langle x|-p\rangle$ ✓

$= |-p\rangle$ ✓

Now we have $P|p\rangle = |-p\rangle$

Since $P = P^{-1}$, we can write

$P^{-1}|p\rangle = |-p\rangle$

Operating on both sides with $P\hat{p}$

$P\hat{p}P^{-1}|p\rangle = P\hat{p}|-p\rangle$ ✓

$$\mathcal{P}\hat{p}\mathcal{P}^{-1}|p\rangle = -p\,\mathcal{P}|-p\rangle$$

$$= -p\,|\hat{p}\rangle$$

$$= -\hat{p}\,|p\rangle \checkmark$$

i.e

$$\mathcal{P}\hat{p}\mathcal{P}^{-1}|p\rangle + \hat{p}|p\rangle = 0$$

$$\left(\mathcal{P}\hat{p}\mathcal{P}^{-1} + \hat{p}\right)|p\rangle = 0 \checkmark$$

$$\mathcal{P}\hat{p}\mathcal{P}^{-1} = -\hat{p} \quad \checkmark \text{ Proved.}$$

Now

$$\langle\psi|\,\mathcal{P}V(\hat{x})\mathcal{P}^{-1}|\mathcal{g}\rangle = \int dx'\,\langle\psi|x'\rangle\langle x'|\mathcal{P}\,V(\hat{x})\mathcal{P}^{-1}|\mathcal{g}\rangle \text{ ok.}$$

$$= \int dx'\,\langle\psi|x'\rangle\langle -x'|V(\hat{x})\mathcal{P}^{-1}|\mathcal{g}\rangle \checkmark$$

Now

$$V(\hat{x}) = \sum_x V(x)\,|x\rangle\langle x| \checkmark$$

Using it we get

$$= \int dx'\,\langle\psi|x'\rangle\langle -x'|\sum_x V(x)\,|x\rangle\langle x|\mathcal{P}^{-1}|\mathcal{g}\rangle$$

$$= \int dx' \langle \psi | x' \rangle \langle -x' | \sum_x V(x) |x\rangle\langle -x| \varphi \rangle$$ ✓

? Substitute $x \to -x$ / x' ?

$$= \int dx' \langle \psi | x' \rangle \langle x' | \sum_x V(x) |x\rangle\langle -x| \varphi \rangle$$

$$= \int dx' \langle \psi | x' \rangle \langle x' | V(-\hat{x}) | \varphi \rangle$$

$\int \delta(x-x') dx = 1$ picks up $x = x'$

$$= \langle \psi | V(-\hat{x}) | \varphi \rangle$$

$$\Rightarrow \quad P V(\hat{x}) P^{-1} = V(-\hat{x})$$ ✓ since

Now $P \hat{H} P^{-1} = P \frac{\hat{p}^2}{2m} + V(\hat{x}) P^{-1} = P \frac{\hat{p}^2}{2m} P^{-1} + P V(\hat{x}) P^{-1}$

Hamiltonian under parity. $\Longrightarrow$

$= \frac{1}{2m} P \hat{p} \hat{p} P^{-1} + P V(\hat{x}) P^{-1} = \frac{1}{2m} P \hat{p} P^{-1} P \hat{p} P^{-1} + P V(x) P^{-1}$ ✓ ($P^{-1}P = 1$ (identity))

$= \frac{1}{2m} (-\hat{p})(-\hat{p}) + V(-\hat{x})$

$= +\frac{\hat{p}^2}{2m} + V(\hat{x})$ since potential is symmetric.

46. Set $\alpha_0 = \sqrt{\frac{-2mV_0}{\hbar}}$.

5/6.

we have $\alpha = \sqrt{\frac{2m(E-V_0)}{\hbar}}$ ✓ $\Rightarrow \alpha^2 = \frac{2m(E-V_0)}{\hbar^2}$

$$\beta^2 = \frac{-2mE}{\hbar^2}$$ ✓

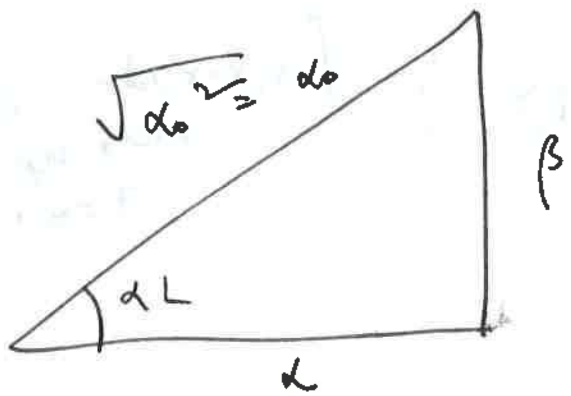
Now, we recognize that ok

$$\alpha^2 = -\beta^2 + \alpha_0^2$$ ✓.

$$\Rightarrow \quad \alpha^2 + \beta^2 = \alpha_0^2$$

Let's take a right angled Δ

where α_0 is the hypotenuse



$$\tan(\alpha L) = \frac{\beta}{\alpha}$$ ok

$$\cos(\alpha L)\Big|_{\text{from diagram}} = \frac{\alpha}{\alpha_0}$$ ✓

?

$$\cos^2(\alpha L) = \frac{\alpha^2}{\alpha_0^2}$$ ✓

$$|\cos(\alpha L)| = \frac{\alpha}{\alpha_0}$$ ✓ → modulus value.

$\tan \alpha L > 0$?

This is for the even state ..

Now, for the odd state

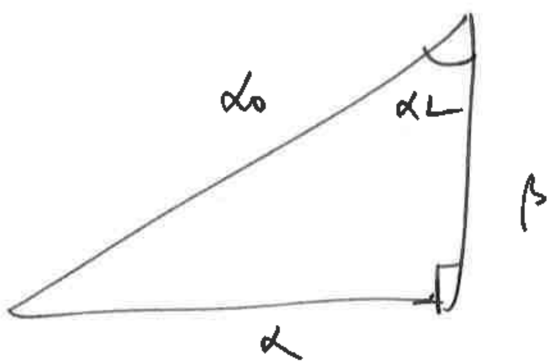
$$\cot(\alpha L) = \frac{-\beta}{\alpha}$$

Taking square

$$\cot^2(\alpha L) = \frac{\beta^2}{\alpha^2}$$

$$\Rightarrow \quad \tan^2(\alpha L) = \frac{\alpha^2}{\beta^2}$$

Now, we see that angle changes ✓



$$\sin(\alpha L) = \frac{\alpha}{\alpha_0}$$

$$\sin^2(\alpha L) = \left(\frac{\alpha}{\alpha_0}\right)^2$$

$$|\sin(\alpha L)| = \left(\frac{\alpha}{\alpha_0}\right)$$ ✓

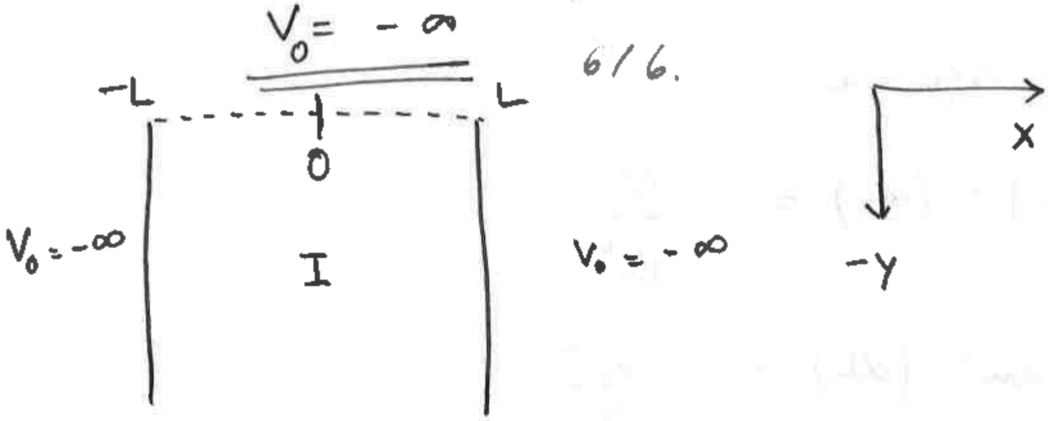
✓

NICE but could have been a bit more complete.

Näive but effective way!!

47. The wavefunction at $|x| \geq L$ vanishes since the potential is infinite.

6/6.



Schrödinger equation reads in Region I, i.e $-L \leq x \leq L$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \checkmark$$

$$\frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi \qquad \text{Let } k = \frac{\sqrt{+2mE}}{\hbar}$$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} = -k^2\psi$$

$$\psi(x) = A\sin kx + B\cos kx \checkmark \quad - \text{①}$$

$$\psi(-x) = -A\sin kx + B\cos kx \quad - \text{②}$$

① + ② leaves cos terms

gives $kx = \frac{n\pi}{2} \Rightarrow k = \frac{n\pi}{2x}$

① - ② leaves sin term

$2A\sin(kx) = 0$

$kx = n\pi \Rightarrow k = \frac{n\pi}{x}$

$k = \frac{n\pi}{x}$

Ahh.. $\underline{\underline{x}}$ is $\underline{\underline{L}}$ in this case.

Rewriting ..

$V_0 = -\infty$ Well

COSINE

$$k = \frac{n\pi}{2L}$$

where $n = 1, 2, 3$ ✓

SINE

$$k = \frac{n\pi}{L}$$

where $n = 1, 2, 3$ ✓

$$\boxed{k^2 = \frac{2mE}{\hbar^2}}$$

Rewrite the conditions of SIN as

$$k = \frac{n\pi}{2L} \quad \text{where } n = 2, 4, 6, \ldots$$

we get $\psi_n = A_n \sin\left(\frac{n\pi x}{2L}\right)$ $\quad n = 2, 4, 6$ Parity?

and $\psi_n = B_n \cos\left(\frac{n\pi x}{2L}\right)$ $\quad n = 1, 3, 5, 7$

$$\frac{n\pi}{2L} = k$$

$$\frac{n^2\pi^2}{4L^2} = k^2 \Rightarrow \frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{4L^2}$$

$$\Rightarrow \boxed{E = \frac{n^2\pi^2\hbar^2}{8mL^2}}$$ ✓

Energy eigenvalue

$$\psi_n(x) = \sqrt{\frac{2}{2L}} \sin\left(\frac{n\pi x}{2L}\right),$$

$$= \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right), \quad n = \text{even}$$

$$\psi_n(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{n\pi x}{2L}\right), \quad n = \text{odd}$$

Now $\sin(-x) = -\sin(x)$

$\cos(-x) = \cos(x)$

$$\psi_n(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right), \quad n = \text{even}$$ ✓

→ PARITY ODD

$$\psi_n(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{n\pi x}{2L}\right), \quad n = \text{odd}$$ ✓

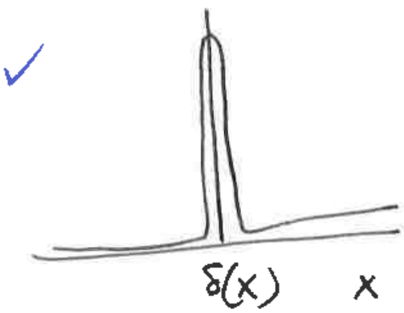
→ PARITY EVEN.

48. 48. $V = V_0\,\delta(x)$ 6/6., $V_0 < 0$

Energy eigenvectors and eigenvalues of all bound states ($E < 0$) ✓

Solution: SE (Schrödinger equation) is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\,\delta(x)\,\psi = E\psi \quad ✓$$



Look at bound states

In the region $x<0$, $V(x)=0$, so

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \quad ✓$$

$$= k^2\psi \qquad \text{— (1)} \qquad \text{where } k = \sqrt{\frac{-2mE}{\hbar}}$$

E is negative, k is real and positive

General solution

$$\psi(x) = Ae^{-kx} + Be^{-kx} \quad ✓$$

first term blows up as $x \to -\infty$, so $A = 0$

$$\boxed{\psi(x) = Be^{-kx}} \quad ✓ \qquad x < 0$$

In the region $x > 0$ $V(x) = 0$

we have $\psi(x) = Fe^{-kx} + Ge^{kx}$

here $G = 0$, since $x \to \infty$ need to stop blowing

$\psi(x) = F e^{-kx}$ ✓ $(x > 0)$

Boundary conditions

1st $\longrightarrow$ ψ continuous ✓

2nd $\longrightarrow$ $\frac{d\psi}{dx}$ is continuous except at 0, where potential blows up ✓

The first condition gives at $\underset{x \to 0}{Lt}$, $F = B$

Delta function determines the discontinuity in the derivative at $x = 0$

Integrate Schrödinger Equation from $-\epsilon$ to ϵ and let $\underset{\epsilon \to 0}{Lt}$ ✓

$$-\frac{\hbar^2}{2m} \int_{-\epsilon}^{+\epsilon} \frac{d^2\psi}{dx^2} dx + \int_{-\epsilon}^{\epsilon} V(x)\psi(x)\, dx = E \int_{-\epsilon}^{\epsilon} \psi(x)\, dx \rightarrow 0$$

The R.H.S goes to zero in the limit

$$\Delta\left(\frac{d\psi}{dx}\right) = \frac{2m}{\hbar^2} \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} V(x)\psi(x)\,dx$$

$$= \frac{2m}{\hbar^2} \int_{-\epsilon}^{\epsilon} V_0\, \delta(x)\psi(x)\,dx$$

$$= \frac{2mV_0}{\hbar^2}\psi(0)$$

we had, the solutions, take derivative

$$\frac{d\psi}{dx} = -BKe^{-Kx} \quad \text{for } x>0 \quad \left.\frac{d\psi}{dx}\right|_+ = -BK$$

$$\frac{d\psi}{dx} = BKe^{Kx} \quad \text{for } x<0 \quad \left.\frac{d\psi}{dx}\right|_- = BK$$

Also $\boxed{\psi(0) = B}$ setting $x=0$ in our solution

$$\Delta\left(\frac{d\psi}{dx}\right) = -2BK \quad ✓ \quad ; \quad \psi(0) = B$$

$$-\cancel{2}\cancel{B}K = \frac{\cancel{2}mV_0}{\hbar^2}\cancel{B}$$

$$K = \frac{mV_0}{\hbar^2} \quad ✓$$

$$E = \frac{-\hbar^2 k^2}{2m} = \frac{-mV_0^2}{2\hbar^2} \quad ✓$$

→ Energy eigenvalue

$\psi(x) = B e^{-kx}$

Normalize no need

$$\int_{\infty}^{\infty} |\psi(x)|^2 dx = 2B^2 \int_0^{\infty} e^{-2kx} dx$$

$$= \frac{|B|^2}{k} = 1$$

$$\Rightarrow B = \sqrt{k} = \sqrt{\frac{m V_0}{\hbar}}$$

$$\psi(x) = \sqrt{\frac{m V_0}{\hbar}} \, e^{+\frac{m V_0 |x|}{\hbar^2}}$$ ✓

Bound state ✓
energy eigenfunction

49.

49. i) $S = \dfrac{2p\bar{p}}{2p\bar{p}\cos\left(\frac{2\bar{p}L}{\hbar}\right) - i(p^2+\bar{p}^2)\sin\left(\frac{2\bar{p}L}{\hbar}\right)}$

49.
4/4.

$$|S|^2 = \frac{4p^2\bar{p}^2}{4p^2\bar{p}^2\cos^2\left(\frac{2\bar{p}L}{\hbar}\right) + (p^2+\bar{p}^2)^2\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)} \quad - (1)$$

✓

Now $Q = \dfrac{i(\bar{p}^2-p^2)\sin\left(\frac{2\bar{p}L}{\hbar}\right)}{2p\bar{p}\cos\left(\frac{2\bar{p}L}{\hbar}\right) - i(\bar{p}^2+p^2)\sin\left(\frac{2\bar{p}L}{\hbar}\right)}$

$$|Q|^2 = \frac{(\bar{p}^2-p^2)^2\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)}{4p^2\bar{p}^2\cos^2\left(\frac{2\bar{p}L}{\hbar}\right) + (p^2+\bar{p}^2)^2\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)} \quad - (2)$$

✓

Adding (1) & (2)

$$|S|^2+|Q|^2 = \frac{4p^2\bar{p}^2 + \left[\bar{p}^4 + p^4 - 2\bar{p}^2p^2\right]\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)}{4p^2\bar{p}^2\cos^2\left(\frac{2\bar{p}L}{\hbar}\right) + (p^2+\bar{p}^2)^2\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)}$$

$$= \frac{4p^2\bar{p}^2 + \left\{\left[\bar{p}^2+p^2\right]^2 - 4\bar{p}^2p^2\right\}\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)}{4p^2\bar{p}^2\cos^2\left(\frac{2\bar{p}L}{\hbar}\right) + (p^2+\bar{p}^2)\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)}$$

$$= \frac{4p^2\bar{p}^2\left[1-\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)\right] + (\bar{p}^2+p^2)\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)}{4p^2\bar{p}^2\cos^2\left(\frac{2\bar{p}L}{\hbar}\right) + (\bar{p}^2+p^2)\sin^2\left(\frac{2\bar{p}L}{\hbar}\right)}$$

✓

$$= 1 .$$

×

49. ii $P|p\rangle = \int dx \; P|x\rangle\langle x|p\rangle$

$$= \int dx \; |-x\rangle\langle x|p\rangle$$

$$= \int dx \; |x\rangle\langle -x|p\rangle$$

$$\langle -x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ipx}{\hbar}} = \langle x|-p\rangle$$

$$P|p\rangle = \int dx \; |x\rangle\langle x|-p\rangle$$

$PP^+ = \mathbb{1}$

$P^+ = P^{-1} = P$

P^+ $\quad P|p\rangle = |-p\rangle$

$\langle p|P^+ = \langle -p|$

$\langle p|P\,p\,P^{-1}|p\rangle =$

$\langle p|P\,p\,P^+|p\rangle = \langle -p|\,p\,|-p\rangle$

$= \langle p|\,p$

49.6 ii) Poles of $S(E)$

$$S = \frac{2p\tilde{p}}{2p\tilde{p}\cos\left(\frac{2\tilde{p}L}{\hbar}\right) - i\left(\tilde{p}^2 + p^2\right)\sin\left(\frac{2\tilde{p}L}{\hbar}\right)} e^{\frac{-2ipL}{\hbar}}$$

Poles are when the denominator goes to zero. ✓

That happens when,

$$2p\tilde{p}\cos\left(\frac{2\tilde{p}L}{\hbar}\right) = i\left(\tilde{p}^2 + p^2\right)\sin\left(\frac{2\tilde{p}L}{\hbar}\right) \quad ✓$$

$$\Rightarrow \cot\left(\frac{2\tilde{p}L}{\hbar}\right) = \frac{i\left(\tilde{p}^2 + p^2\right)}{2p\tilde{p}}$$

Now, lets make a substitution

$$p \rightarrow i\xi$$

OK.

we get,

$$\cot\left(\frac{2\tilde{p}L}{\hbar}\right) = \frac{i\left(\tilde{p}^2 - \xi^2\right)}{i\,2\tilde{p}\xi} \quad ✓$$

$$= \frac{\left(\tilde{p}^2 - \xi^2\right)}{2\tilde{p}\xi}$$

$$\frac{1-\tan^2\left(\frac{\tilde{p}L}{\hbar}\right)}{2\tan\left(\frac{\tilde{p}L}{\hbar}\right)} = \frac{\tilde{p}^2-\xi^2}{2\tilde{p}\xi}$$

Solving

$$\Rightarrow \tan\left(\frac{\tilde{p}L}{\hbar}\right) = \frac{-(\tilde{p}^2-\xi^2) \pm \sqrt{(\tilde{p}^2-\xi^2)^2+4\tilde{p}^2\xi^2}}{2\tilde{p}\xi}$$

$$= \frac{\xi^2 \pm \xi^2 - \tilde{p}^2 \pm \tilde{p}^2}{2\tilde{p}\xi}$$

$$\Rightarrow \tan\left(\frac{\tilde{p}L}{\hbar}\right) = \frac{\xi}{\tilde{p}}$$ even states

and $$\cot\left(\frac{\tilde{p}L}{\hbar}\right) = -\frac{\xi}{\tilde{p}}$$ odd states

Now, this is exactly the points of bound state. ✓

Bound state solution.

Zeros of denominator ⇒ Poles of S(E) ⟹ Bound states !! ✓

50. 50. $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}$ 4.4.

we have $\vec{j}(t,\vec{x}) = \frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$

$$-\vec{\nabla}\cdot\vec{j} = -\vec{\nabla}\cdot\left(\frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right)$$

$$= \frac{-\hbar}{2mi}\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$= \frac{\hbar}{2mi}\nabla\cdot\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$$

$$= \frac{\hbar}{2mi}\left[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right] ✓$$

$$= \frac{1}{i\hbar}\cdot\frac{\hbar^2}{2m}\left[\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right] ✓$$

$$= \frac{1}{i\hbar}\left[\frac{\hbar^2}{2m}\psi\nabla^2\psi^* - \frac{\hbar^2}{2m}\psi^*\nabla^2\psi - V(x)\psi^*\psi + V(x)\psi\psi^*\right]$$

↳ Adding and subtracting this

$$= \frac{1}{i\hbar}\left[\frac{\hbar^2}{2m}\nabla^2\psi^*\psi - V(x)\psi^*\psi - \left\{\frac{\hbar^2}{2m}\nabla^2\psi\psi^* - V(x)\psi\psi^*\right\}\right] ✓$$

Now using Schrödinger equation

$$= \frac{1}{i\hbar}\left[+i\hbar \frac{\partial \psi^*}{\partial t}\psi - \left(-i\hbar \frac{\partial \psi}{\partial t}\psi^*\right)\right]$$

$$= \frac{1}{i\hbar}\left[i\hbar \frac{\partial \psi}{\partial t}\psi^* + i\hbar \frac{\partial \psi^*}{\partial t}\psi\right]$$ ✓

$$= \left[\frac{\partial \psi}{\partial t}\psi^* + \frac{\partial \psi^*}{\partial t}\psi\right]$$ ✓

$$= \frac{\partial}{\partial t}(\psi\psi^*)$$ ✓

$$= \frac{\partial \rho}{\partial t}$$

Hence, we have $\vec{\nabla}\cdot\vec{j} = \frac{-\partial \rho}{\partial t}$ ✓

we have used the fact that:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = i\hbar\frac{\partial \psi}{\partial t}$$ ✓

and $$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*(x)}{\partial x^2} + V(x)\psi^*(x) = -i\hbar\frac{\partial \psi^*(x,t)}{\partial t}$$ ✓

NOTE: This was using the fact that we know $\vec{j}$ ok.

50. 50.

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial t}\left(\psi^*(x,t)\psi(x,t)\right)$$

Second way

$$= \left[\frac{\partial \psi^*}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t}\right]$$

This is how we convince that $\vec{j}$ should be what it is.

$$= \frac{1}{i\hbar}\left[i\hbar\frac{\partial \psi^*}{\partial t}\psi + i\hbar\psi^*\frac{\partial \psi}{\partial t}\right]$$

$$= \frac{1}{i\hbar}\left[\frac{\hbar^2}{2m}\nabla^2\psi^*\psi - V(x)\psi^*\psi + \left(-\frac{\hbar^2}{2m}\right)\psi^*\nabla^2\psi + V(x)\psi^*\psi\right]$$

$$= \frac{\hbar^2}{2m}\frac{1}{i\hbar}\left[\frac{\partial^2\psi^*}{\partial x^2}\psi - \psi^*\nabla^2\psi\right]$$

$$= \frac{\hbar}{2mi}\frac{\partial}{\partial x}\left[\frac{\partial\psi^*}{\partial x}\psi - \psi^*\frac{\partial\psi}{\partial x}\right]$$

$$= -\frac{\partial}{\partial x}\left[\frac{\hbar}{2mi}\left(\psi\nabla\psi^* + \psi^*\nabla\psi\right)\right]$$

Same.

$$= -\nabla\cdot\vec{j}$$

Identify $\vec{j}(\bar{x},t) = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$

THE END ✓

44/48

5. We have from lecture

a.

$$Q = \frac{i(\tilde{p}^2 - p^2)\sin\left(\frac{2\tilde{p}L}{\hbar}\right)}{2p\tilde{p}\cos\left(\frac{2\tilde{p}L}{\hbar}\right) - i(\tilde{p}^2 + p^2)\sin\left(\frac{2\tilde{p}L}{\hbar}\right)} e^{-\frac{2ipL}{\hbar}}$$

8/8

Substitute $\tilde{p} \to i\tilde{p}'$

$$Q(E) = \frac{i(-\tilde{p}'^2 - p^2)\sin\left(\frac{2i\tilde{p}'L}{\hbar}\right)}{2i\tilde{p}'p\cos\left(\frac{2i\tilde{p}L}{\hbar}\right) - i(p^2 - \tilde{p}^2)\sin\left(\frac{2i\tilde{p}'L}{\hbar}\right)} e^{-\frac{2ipL}{\hbar}}$$

$\sin(iz) = i\sinh z$
and $\cos(iz) = \cosh z$ } use in above

$$Q(E) = \frac{(\tilde{p}'^2 + p^2)\sinh\left(\frac{2\tilde{p}L}{\hbar}\right)}{2i\tilde{p}'p\cosh\left(\frac{2i\tilde{p}L}{\hbar}\right) + (p^2 - \tilde{p}^2)\sinh\left(\frac{2\tilde{p}L}{\hbar}\right)} e^{-\frac{2ipL}{\hbar}}$$

S(E) given in lecture notes, squaring SS^*

b.

$$|S(E)|^2 = \frac{+4p^2\tilde{p}'^2}{(p^2 - \tilde{p}'^2)^2\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + 4p^2\tilde{p}^2\cosh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$|Q(E)|^2 = \frac{(\tilde{p}'^2 + p^2)^2\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}{(p^2 - \tilde{p}'^2)^2\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + 4p^2\tilde{p}'^2\cosh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

Adding we get

$$|S(E)|^2 + |Q(E)|^2 = \frac{(\tilde{p}'^4 + p^4 + 2\tilde{p}'^2 p^2 - 2\tilde{p}'^2 p^2)\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + 4p^2\tilde{p}'^2}{(p^2 - \tilde{p}'^2)\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + 4\tilde{p}^2 p^2 \cosh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$= \frac{(\tilde{p}'^2 - p^2)^2 \sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + 4p^2\tilde{p}^2\left(1 + \sinh^2\left(\frac{\tilde{p}L}{\hbar}\right)\right)}{(\tilde{p}'^2 - p^2)^2 \sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + 4p^2\tilde{p}^2 \cosh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$= 1 \checkmark$$

$$\left[1 + \sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) = \cosh^2\left(\frac{2\tilde{p}L}{\hbar}\right)\right] \checkmark$$

c. $$|S(E)|^2 = \frac{4p^2\tilde{p}'^2}{(p^2 - \tilde{p}'^2)^2 \sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + \left(4p^2\tilde{p}^2\left(1 + \sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right)\right)\right)}$$

$$= \frac{4p^2\tilde{p}'^2}{(p^4 + \tilde{p}'^4 - 2p^2\tilde{p}'^2)\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + 4p^2\tilde{p}^2 + 4p^2\tilde{p}^2\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$= \frac{4p^2\tilde{p}'^2}{(p^2 + \tilde{p}^2)^2}$$

c. $$|S(E)|^2 = \frac{(2p\tilde{p})^2}{(p^2-\tilde{p}^2)^2\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right) + (2p\tilde{p})^2\cosh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$= \frac{(2p\tilde{p})^2}{\left[1+\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right)\right](2p\tilde{p})^2 + (p^2-\tilde{p}^2)^2\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$= \frac{(2p\tilde{p})^2}{(2p\tilde{p})^2 + \left((2p\tilde{p})^2+(p^2-\tilde{p}^2)^2\right)\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$= \frac{1}{1+\frac{(p^2+\tilde{p}^2)^2}{(2p\tilde{p})^2}\sinh^2\left(\frac{2\tilde{p}L}{\hbar}\right)} \quad -\ (ii)$$

Now $p = \sqrt{2mE} \Rightarrow p^2 = 2mE$

$\tilde{p} = \sqrt{2m(V_0 - E)} \Rightarrow \tilde{p}^2 = 2m(V_0 - E)$

In view of (ii), lets find out $\frac{(p^2+\tilde{p}^2)^2}{4p^2\tilde{p}^2}$

Also $(p^2+\tilde{p}^2)^2 = (2mV_0)^2$

$4p^2\tilde{p}^2 = 4(2mE)(2m(V_0-E))$

Plugging in (ii)

$$|S|^2 = \frac{1}{1 + \frac{4m^2 V_0^2}{4m^2 (2E)(2V_0 - 2E)} \sin^2 h\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

$$= \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sin^2 h\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

d. $$T(E) = \frac{1}{1 + \underline{\frac{V_0^2}{4E(V_0-E)}} \sin^2 h\left(\frac{2\tilde{p}L}{\hbar}\right)}$$

as $E \to 0$, the underlined term blows up.

$$\lim_{E \to 0} T(E) = 0.$$

Now, as $E \to V_0$

$$\sinh^2\left(\frac{2\tilde{p}'L}{\hbar}\right) = \left(\frac{2\tilde{p}'L}{\hbar}\right)^2$$

for small argument.

$$T(E) = \frac{1}{1 + \frac{V_0^2}{4V_0(V_0 - E)} \sinh^2\left(\frac{2\hat{p}'L}{\hbar}\right)}$$

d. $T(E) = \dfrac{1}{1 + \dfrac{V_0^2}{\cancel{4E}(V_0-E)} \cdot \dfrac{AL^2}{\hbar^2} \cancel{2m(V_0-E)}}$

$= \dfrac{1}{1 + \dfrac{2mL^2 V_0^2}{\hbar^2 E}}$ ✓ ✓.

52. $\hat{a} = \sqrt{\dfrac{m\omega}{2\hbar}}\,\hat{q} + \dfrac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ ✓

4/4

$\hat{a}^+ = \sqrt{\dfrac{m\omega}{\cancel{2\hbar}}}\,\hat{q} - \dfrac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$

we find $[a^+, a]$

Not required

$[a^+, a] = -\dfrac{1}{2\hbar}\left(-i[x,p] + i[p,x]\right)$

$\hat{a}^+ a = \dfrac{m\omega}{2\hbar}\hat{q}^2 + i\dfrac{\hat{q}\hat{p}}{2\hbar} - \dfrac{i}{2\hbar}\hat{p}\hat{q} - \dfrac{i^2}{2m\hbar\omega}\hat{p}^2$

$= \dfrac{m\omega}{2\hbar}\left(\hat{q}^2 + \dfrac{\hat{p}^2}{m^2\omega^2}\right) + \left(\dfrac{i}{2\hbar}\right)[\hat{q}, \hat{p}]$ ✓

$$a^{+}a = \frac{m\omega}{2\hbar}\left(q^2 + \frac{p^2}{m^2\omega^2}\right) + \frac{i}{2\hbar}[q, p]$$

$$= \frac{1}{\hbar\omega}\left(\frac{1}{2}m\omega^2\hat{q}^2 + \frac{\hat{p}^2}{2m}\right) + \frac{i}{2\hbar}[q, p]$$

we identify $\boxed{\frac{1}{2}m\omega^2\hat{q}^2 + \frac{\hat{p}^2}{2m} = \hat{H}}$ ✓

$$= \frac{H}{\hbar\omega} + \frac{i}{2\hbar} i\hbar$$

$$= \frac{H}{\hbar\omega} - \frac{1}{2}$$ ✓

But $a^{+}a = N$ ✓

$$\boxed{\hbar\omega\left(N + \frac{1}{2}\right) = H}$$ ✓ Proved.

53. i). $[a, a^{+}] = 1$

318

$[a, a^{+}] = aa^{+} - a^{+}a$

$$= \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{i}{\sqrt{2m\hbar\omega}}\hat{x}\hat{p} + i$$

53. i.
(4)

$$aa^\dagger = \frac{m\omega}{2\hbar}\hat{x}^2 - \frac{i}{\sqrt{\cancel{2m\omega\hbar}}\,2\hbar}\hat{x}\hat{p} + \frac{i}{2\hbar}\hat{p}\hat{x} + \frac{\hat{p}^2}{2m\hbar\omega} \quad - (1)$$

$$-a^\dagger a = \frac{-1}{\hbar\omega}\left(\frac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m}\right) - \frac{i}{2\hbar}[x,p] \quad - (2)$$

using $a^\dagger a$ from <u>52 Exercise</u> and also using $\hat{a}^\dagger$ & $\hat{a}$.

$$[a,a^\dagger] = aa^\dagger - a^\dagger a$$

$$= \frac{1}{2\hbar}\left(-i[x,p] + i[p,x]\right)$$

$$= \frac{1}{2\hbar}\left(-i\ i\hbar \quad + i\ -i\hbar\right)$$

$$= 1.$$

✓

53. ii) $[N,a] = -a$

Here $N = a^\dagger a$

$$[a^\dagger a, a] = a^\dagger[a,a] + [a^\dagger,a]a \qquad ([a,a] \to 0)$$

$$= -a$$ ✓. [use $[a^\dagger,a] = -1$ from PART 1.

53. iii. $[N, a^\dagger] = +a^\dagger$

$$[a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] + [a^\dagger, a^\dagger]a \quad ([a^\dagger, a^\dagger] \to 0)$$

$$= a^\dagger[a, a^\dagger]$$

$$= +a^\dagger \quad \checkmark \qquad [\text{use } [a, a^\dagger] = +1]$$

$$= a^\dagger \quad \checkmark$$

53. iv. Prove $\langle\psi|N|\psi\rangle \geq 0$

Invoke $N = a^\dagger a$

$$\langle\psi|a^\dagger a|\psi\rangle$$

Now consider the definition of adjoint of an operator (NOT HERMITIAN THOUGH)

$$\langle\psi|a^\dagger a|\psi\rangle \equiv \langle a\psi|a\psi\rangle$$

$$= |\langle a\psi\rangle|^2 \geq 0$$

ALWAYS..

$(\text{Norm})^2 \geq 0$

Hence, we have $\langle\psi|N|\psi\rangle \geq 0$ ✓

53. iv. Hence <u>N is positive</u>.

<u>54. i.</u> Let's prove a result that will be useful in our problems.

6/8

$$[\hat{a}, (\hat{a}^{+})^{n}] = n(\hat{a}^{+})^{n-1}$$ ✓

Lets see for $n=0$

$$[\hat{a}, 1] = 0$$ ✓. ✓ <u>OKAY</u>.

For $n=1$

$$[a, \hat{a}^{+}] = n(\hat{a}^{+})^{1-1}$$

OK

$$= 1$$ [which we proved last exercise]

Let us assume for $\underline{n}$. we'll prove for $n+1$

$$[a(\hat{a}^{+})^{n+1}] = [a, (a^{+})^{n}]\hat{a}$$

$$= [n(\hat{a}^{+})^{n-1}]\hat{a}$$

$$= (n+1)(\hat{a}^{+})^{n}$$ ✓ — (A)

using

$$[a, bc] = b[a,c] + [a,b]c$$

$$[a, \hat{a}^{+n+1}] = [a, a^{+}a^{+n}]$$

$$= a^{+} n a^{+n-1} + [a, a^{+}]a^{+n}$$

$$= (n+1)(\hat{a}^{+})^{n}$$

<u>54. i</u>

Now let's do

$$N|n\rangle = a^{+}a \frac{1}{\sqrt{n!}} (a^{+})^{n}|0\rangle$$

$$N|n\rangle = \frac{1}{\sqrt{n!}} a^{+} a \, (a^{+})^{n} |0\rangle$$

$$= \frac{1}{\sqrt{n!}} a^{+} (a^{+})^{n} a |0\rangle + \frac{1}{\sqrt{n!}} a^{+} \, n \, (a^{+})^{n-1} |0\rangle$$

↓ ZERO as $a|0\rangle = \vec{0}$ → we used Ⓐ

$$= \frac{1}{\sqrt{n!}} n \, (a^{+})^{n} |0\rangle$$

$$= n |n\rangle$$ ✓

54. ii) $|n\rangle = \frac{1}{\sqrt{n!}} (a^{+})^{n} |0\rangle$

$\langle n| = \frac{1}{\sqrt{n!}}$ [redacted] $\langle 0|(a)^{n}$

$$\langle n|n\rangle = \frac{1}{n!} \langle 0| (a a^{+})^{n} |0\rangle$$

$$= \frac{1}{n!} \langle 0| (a^{+})^{n} a |0\rangle + \frac{1}{n!} \langle 0| n (a^{+})^{n-1} |0\rangle$$

$\left[(a a^{+})^{n} = n (a^{+})^{n-1} + (a^{+})^{n} a \right]$

=

$$\langle n|n\rangle = \frac{1}{n!} \langle 0| a^{n} (a^{+})^{n} |0\rangle$$ ✓

$$= \frac{1}{n!} \langle 0| a^{n} \sqrt{n!} |n\rangle = \frac{1}{\sqrt{n!}} \langle 0| a^{n} |n\rangle$$

54. ii) $\langle n|n\rangle = \frac{1}{\sqrt{n!}} \langle 0|a^n|n\rangle$

← But $a^n|n\rangle = \sqrt{n!}\,|0\rangle$

why?

$$= \frac{1}{\sqrt{n!}} \sqrt{n!}\,\langle 0|0\rangle$$

$$= 1.$$

54. iii) $a|n\rangle = \sqrt{n}\,|n-1\rangle$

As discussed in lecture in class

$$a|n\rangle \propto |n-1\rangle$$

Let $a|n\rangle = c\,|n-1\rangle$ — (1)

We'll assume that $|n\rangle$ and $|n-1\rangle$ are normalized.

Taking adjoint of (1) gives

$$\langle n|a^{\dagger} = c^{*}\langle n-1| \quad — (2)$$

From (1) & (2)

$$\langle n|a^{\dagger}a|n\rangle = cc^{*}\langle n-1|n-1\rangle$$

$$= |c|^2$$

We have $n = |c|^2$

'c' to be real and positive $\Rightarrow c = \sqrt{n}$

we have

$$a|n\rangle = c|n-1\rangle$$
$$= \sqrt{n}\,|n-1\rangle$$

~~54.iii AGAIN:~~

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle$$

$$a|n\rangle = \frac{1}{\sqrt{n!}} a (a^+)^n |0\rangle$$
$$= \frac{1}{\sqrt{n!}} a \sqrt{n!}\,|n\rangle$$
$$= \sqrt{n}\,|n-1\rangle$$

Second Method.

54.iii $N a|n\rangle = (n-1)\, a|n\rangle$ Given in notes,

$$= (n-1)\, a \frac{(a^+)^n |0\rangle}{\sqrt{n!}}$$
$$= \frac{(n-1)}{\sqrt{n!}} a a^+ (a^+)^{n-1} |0\rangle$$
$$= \frac{(n-1)}{\sqrt{n!}} (1 + a^+ a) \sqrt{(n-1)!}\,|n-1\rangle$$

$\{ aa^+ = 1 + a^+a$

$$= \frac{n-1}{\sqrt{n}} \left[|n-1\rangle + (n-1)|n-1\rangle \right]$$
$$= \frac{n-1}{\sqrt{n}} (n - \not{1} + \not{1})\, |n-1\rangle$$

$(n-1)\sqrt{n}\,|n-1\rangle$

$\Rightarrow$ $\boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$ ✓

54. iv. $N a^{\dagger}|n\rangle = (n+1)\, a^{\dagger}|n\rangle$ [In notes] — ①

$= (n+1)\, a^{\dagger}\left(\frac{1}{\sqrt{n!}} (a^{\dagger})^{n}|0\rangle\right)$

$= \frac{n+1}{\sqrt{n!}} (a^{\dagger})^{n+1}|0\rangle$

$= \frac{n+1}{\sqrt{n!}} \sqrt{n!}\,\sqrt{n+1}\,|n+1\rangle$

$= n+1\, \sqrt{n+1}\,|n+1\rangle$ — ②

$\Rightarrow$ $\boxed{a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle}$ using ① & ②

(B)

55.a. $\psi_n(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right) e^{-\frac{m\omega}{2\hbar}q^2}$ — (1)

55. 6/8

Let $\sqrt{\frac{m\omega}{\hbar}}\, q = x$

$\left(\frac{m\omega}{\hbar}\right) q^2 = x^2$ — (2)

ok

and $\boxed{\frac{d^2}{dq^2} = \frac{m\omega}{\hbar}\frac{d}{dx^2}}$

Recall Schrödinger equation,

$$\left[-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 \frac{\hbar}{m\omega}x^2\right]\psi(x) = E\psi(x)$$

$$\hbar\omega\left[-\frac{1}{2}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}x^2\psi(x)\right] = E\psi(x) \quad \text{— (3)}$$

Now using (1) & (2), we have

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}} H_n(x)e^{-x^2/2}$$

Let $A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}$

$$\Rightarrow \quad \psi(x) = A\, H_n(x)\, e^{-x^2/2}$$

$$\frac{d^2\psi}{dx^2} = A\left[H_n''(x) - 2x H_n'(x)e^{-x^2/2} + H_n(x)\left[(x^2-1)e^{-x^2/2}\right]\right]$$

Using (3) now

$$\hbar\omega\left[-\frac{1}{2}A\left[H_n''(x) - 2xH_n'(x)e^{-x^2/2} + H_n(x)(x^2-1) + \frac{1}{2}x^2 A H_n(x) e^{-x^2/2}\right]\right.$$

$$= E A H_n(x) e^{-x^2/2}$$

$$\hbar\omega\left[-\frac{1}{2}H_n''(x) + xH_n'(x) - \frac{1}{2}x^2 H_n(x) + \frac{1}{2}H_n(x) + \frac{1}{2}x^2 H_n(x)\right] = E H_n(x)$$

$$\hbar\omega\left[xH_n'(x) - \frac{1}{2}H_n''(x) + \frac{1}{2}H_n(x)\right] = E H_n(x)$$

use $H_n'(x) = 2n H_{n-1}(x)$ ✓

and $H_n''(x) = 4n(n-1)H_{n-2}$ ✓

and simplifying ..

$$\hbar\omega\left[2n\left(xH_{n-1} - (n-1)H_{n-2}\right) + \frac{1}{2}H_n(x)\right] = E H_n(x)$$

Now use

$$2x H_{n-1}(x) = H_n + 2(n-1)H_{n-2}(x)$$ ✓

$$\hbar\omega\left[2n\left(\frac{H_n(x)}{2n}\right) + \frac{1}{2}H_n(x)\right] = E H_n(x)$$

$$\hbar\omega\left[n + \frac{1}{2}\right]H_n(x) = E H_n(x)$$

$$\Rightarrow \quad E = \left(n + \frac{1}{2}\right)\hbar\omega \quad \text{for } n = 0, 1, 2, \ldots$$ ✓

55.6

now,

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(x)\, e^{-x^2/2}$$

$H_n(x) =$ { even function, when $n =$ ~~ODD~~ EVEN ✓; odd function, when $n =$ ODD ✓ }

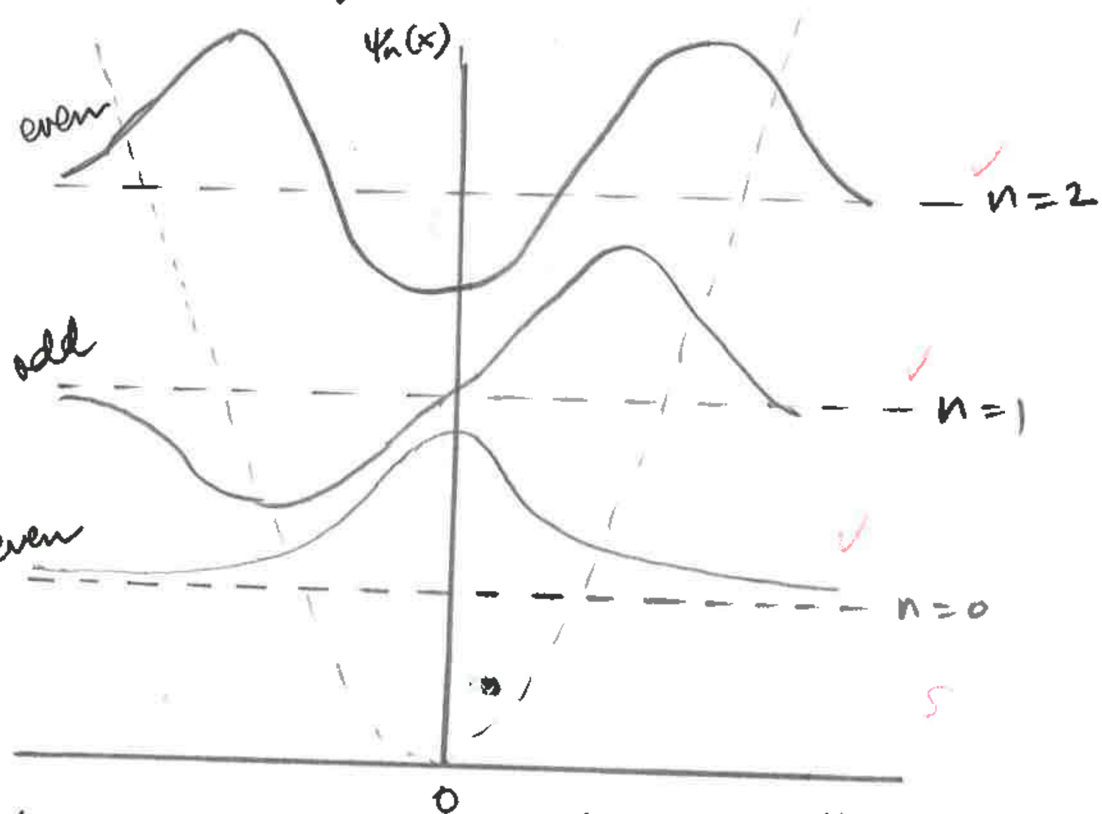
For $n = 0, 2, 4 \ldots$

$\psi_n(x) =$ states of even parity

whereas for $n = 1, 3, 5$

$\psi_n(x)$ is of odd parity.

55.c



56.d. Roots correspond to no. of time sign changes. Then there is no degeneracy.

? Sturm's theorem ?

56.

Coherent state given by

$$|\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \exp(\alpha \hat{a}^{+})|0\rangle$$

Show that this satisfy $a|\alpha\rangle = \alpha|\alpha\rangle$

$$\rightarrow |\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_n \frac{\alpha^n (\hat{a}^{+})^n |0\rangle}{n!}$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n |n\rangle}{\sqrt{n!}}$$

Now let's act with a on $|\psi\rangle$

$$a|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} a|n\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} \sqrt{n}\, |n-1\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= e^{\frac{-|\alpha|^2}{2}} \sum \frac{\alpha^{n+1}}{\sqrt{(n\,\cdot)!}} |n\rangle \quad *$$

Let's do $n \to n+1$ and, since anyways $a|0\rangle = \vec{0}$ we write:

$$= e^{-\frac{|\alpha|^2}{2}} \sum \frac{\alpha^{n+1} \sqrt{n!}}{\sqrt{n!}\,\sqrt{(n\,\cdot)!}} |n\rangle \quad \leftarrow$$

since $n \to n+1$, $\sqrt{(n-1)!}$ becomes $\sqrt{n!}$

$$= \alpha e^{-\frac{|\alpha|^2}{2}} \sum \frac{\alpha^n (a^\dagger)^n |0\rangle}{\sqrt{n!}\sqrt{n!}}$$

$$= \alpha\, \underbrace{e^{-\frac{|\alpha|^2}{2}} \sum \frac{\alpha^n (a^\dagger)^n}{n!} |0\rangle}_{|\alpha\rangle}$$

$$= \alpha |\alpha\rangle$$

57. Prove that coherent states satisfy $\Delta x \Delta p = \hbar/2$

Sol^n: Rewriting $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ & $\hat{a}^+$

4/4 ✓

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^+\right) \; ; \; \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a} - \hat{a}^+\right)$$

Let us now calculate

Coherent State

$$\langle \alpha | x | \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | \hat{a} + \hat{a}^+ | \alpha \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha^* + \alpha\right) \quad -① ✓$$

Since $a|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle \alpha | a^+ = \langle \alpha | \alpha^*$

Similarly, we calculate

$$\langle \alpha | p | \alpha \rangle = -i\sqrt{\frac{m\hbar\omega}{2}} \langle \alpha | \hat{a} - \hat{a}^+ | \alpha \rangle$$

→ SIMILAR ARGUMENT AS ABOVE!

$$= -i\sqrt{\frac{m\hbar\omega}{2}} \left(\alpha - \alpha^*\right) \quad -②$$

$$\langle \alpha | x^2 | \alpha \rangle = \frac{\hbar}{2m\omega} \langle \alpha | (a + a^+)^2 | \alpha \rangle$$

$$= \frac{\hbar}{2m\omega} \left(\langle \alpha | a^2 + (a^+)^2 + a^+ a + a a^+ | \alpha \rangle \right)$$

$$= \frac{\hbar}{2m\omega} \langle \alpha | a^2 + (a^+)^2 + (a^+ a + 1) + a^+ a | \alpha \rangle \quad -③$$

$$= \frac{\hbar}{2m\omega} \left(f^2 + f^{*2} + (f^* f + 1) + f^* f \right) ✓$$

$$= \frac{\hbar}{2m\omega} \left(\alpha^2 + \alpha^{*2} + (\alpha^* \alpha + 1) + \alpha^* \alpha \right)$$ using $f = \alpha$

OK

$$\langle\alpha|p^2|\alpha\rangle = -\frac{m\hbar\omega}{2}\langle\psi|(a-a^\dagger)^2|\psi\rangle$$

$$= -\frac{m\hbar\omega}{2}\langle\psi|\hat{a}^2+(a^\dagger)^2-aa^\dagger-a^\dagger a|\psi\rangle$$

$$= -\frac{m\hbar\omega}{2}\left(\alpha^2+\alpha^{*2}-(\alpha^*\alpha+1)-\alpha^*\alpha\right) \quad \text{— (4)}$$

$$(\Delta x)^2 = \langle\alpha|x^2|\alpha\rangle - (\langle\alpha|x|\alpha\rangle)^2$$

$$= \frac{\hbar}{2m\omega}\left(\alpha^2+\alpha^{*2}+(\alpha^*\alpha+1)+\alpha^*\alpha - \alpha^{*2}-\alpha^2-\alpha^*\alpha-\alpha\alpha^*\right)$$

using (1) and (3)

$$= \frac{\hbar}{2m\omega}$$

$$\Rightarrow \boxed{\Delta x = \sqrt{\frac{\hbar}{2m\omega}}} \quad \text{— (1)}$$

$$(\Delta p)^2 = \langle\alpha|p^2|\alpha\rangle - (\langle\alpha|p|\alpha\rangle)^2$$

$$= \frac{m\hbar\omega}{2}$$

using (2) and (4)

$$\boxed{\Delta p = \sqrt{\frac{m\hbar\omega}{2}}}$$

Minimum uncertainty.

$$\boxed{\Delta x \cdot \Delta p = \frac{\hbar}{2}}$$

✓

58 Continued:

(58)

$$e^{-\frac{iH(t-t_0)}{\hbar}}|\alpha\rangle = U(t,t_0)|\alpha\rangle$$

3/4

$$:= e^{-\frac{iH(t-t_0)}{\hbar}} e^{\alpha a^+} e^{\frac{iH(t-t_0)}{\hbar}} e^{-\frac{iH(t-t_0)}{\hbar}} |0\rangle e^{-\frac{|\alpha|^2}{2}}$$

II

$$:= e^{\alpha e^{-\frac{iH(t-t_0)}{\hbar}} a^+ e^{\frac{iH(t-t_0)}{\hbar}}} e^{-\frac{i\hbar\omega(t-t_0)}{2}} |0\rangle \quad e^{-\frac{|\alpha|^2}{2}}$$

eigenvalue eqn

$$= e^{\alpha a^+ e^{-i\omega(t-t_0)}} |0\rangle e^{-\frac{|\alpha|^2}{2}} e^{-\frac{i\omega(t-t_0)}{2}}$$

Apply BCH to the power of exponent

$$= |\alpha_0 e^{-i\omega(t-t_0)}\rangle e^{-\frac{i\omega(t-t_0)}{2}}$$

(59) $L_i = \sum_{j,k} \epsilon_{ijk} X_j P_k$

25/31

R. Jha
10/31/2013

4/5

$$[L_i, L_j] = \left[\sum_{k,l} \epsilon_{ikl} X_k P_l , \sum_{m,n} \epsilon_{jmn} X_m P_n\right]$$

✓ ? how?

$$= \sum_{k,l,m,n} \epsilon_{ikl}\epsilon_{jmn} \left\{ -i\hbar\, \delta_{lm} X_k P_n + i\hbar\, \delta_{kn} X_m P_l \right\}$$

$$= i\hbar \left\{ \sum_{l,m} \left(\sum_k \epsilon_{ikl} \epsilon_{jmk} \right) X_m P_l - \sum_{k,n} \left(\sum_l \epsilon_{ikl} \epsilon_{jln} \right) X_k P_n \right\}$$

Since δ_{lm} is non zero only when $l = m$ and $\delta_{kn} \neq 0$ when $k = n$

$$= i\hbar \left(\sum_{l,m} \left(\delta_{lj} \delta_{im} - \delta_{lm} \delta_{ij} \right) X_m P_l - \sum_{k,n} \left(\delta_{in} \delta_{kj} - \delta_{ij} \delta_{kn} \right) X_k P_n \right\}$$

$$= i\hbar \left(X_i P_j - X_j P_i \right)$$

Since first and fourth sum to zero

why?

$$= i\hbar\, \epsilon_{ijk} L_k$$

Sum of repeated indices intended.

Note :- Here ϵ_{ijk} is Levi-Civita alternating tensor of rank 3. We can raise or lower indices naturally (using metric tensor or delta) i.e $\epsilon_{ijk} = \epsilon_{ij}{}^{k}$ . .

Gives $[\hat{L}_i, \hat{L}_j] = i\hbar\, \epsilon_{ij}{}^{k}\, \hat{L}_k$

* You can raise or lower indices of ϵ_{ijk} in EUCLIDEAN SPACE at will.

−☺☺çV♥`●pт♥J☺☺p F0δ☺

awkward

<u>Traditional awkward Way :</u>

$$[L_x, L_y] = [YP_z - ZP_y,\ ZP_x - XP_z]$$

$$= Y P_x [P_z, Z] =$$

$$= [YP_z, ZP_x] - [YP_z, XP_z] \to 0$$

$$- [ZP_y, ZP_x] \to 0$$

$$+ [ZP_y, XP_z]$$

$$= [YP_z, ZP_x] + [ZP_y, XP_z]$$

$$= Y[P_z, Z]P_x + P_y[Z, P_z]X$$

$$= i\hbar L_z$$

Similarly

$$\left.\begin{aligned} [L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y \end{aligned}\right\}$$

Compact form $[\hat{L}_i, \hat{L}_j] = i\hbar\, \epsilon_{ij}{}^{k}\, \hat{L}_k$ ✓

`▐Çє└31v—☺♠`·☺☺Ç█☺♥É⌡Çδ@16v┬

60. Show that

414. $[\hat{L}_i, \hat{X}^j] = i\hbar \epsilon_i{}^j{}_k \hat{X}^k$

Sol$^{\underline{n}}$ $[\hat{L}_i, \hat{X}^j] = [\epsilon_i{}^{kl} X_k P_l, X^j]$

$$= \epsilon_i{}^{kl} X_k [P_l, X^j]$$

$[AB, C] = A[B, C] + [A, C]B$

Second term with $[X_k, X^j] = 0$

$$= \epsilon_i{}^{kl} X_k \; -i\hbar \, \delta_l{}^j$$

$$= -i\hbar \, \epsilon_i{}^{kl} \hat{X}_k \delta_l{}^j$$

$$= i\hbar \, \epsilon_i{}^{lk} \hat{X}_k \delta_l{}^j \Leftarrow \boxed{\epsilon_i{}^{kl} = -\epsilon_i{}^{lk}}$$

$$= i\hbar \, \epsilon_i{}^{jk} \hat{X}_k$$

$$= i\hbar \, \epsilon_i{}^j{}_k \hat{X}^k$$

→ Raise and lower using metric tensor of Euclidean space or Kronecker delta.

b) $[\hat{L}_i, \hat{P}_j] = [\epsilon_i{}^{kl} \hat{X}_k \hat{P}_l, \hat{P}_j]$

$$= \epsilon_i{}^{kl} [\hat{X}_k, \hat{P}_j] \hat{P}_l$$

using $[AB, C] = A[B, C] + [A, C]B$

$$= \epsilon_i^{\ kl}\, i\hbar\, \delta_{kj}\ \hat{P}_l$$

$$= i\hbar\, \epsilon_i^{\ kl}\, \hat{P}_l\, \delta_{kj}$$

$$= i\hbar\, \epsilon_{ij}^{\ \ l}\, \hat{P}_l$$

Now since 'l' is dummy index here, we have

$$= i\hbar\, \epsilon_{ij}^{\ \ k}\, \hat{P}_k$$

i.e $[\hat{L}_i, \hat{P}_j] = i\hbar\, \epsilon_{ij}^{\ \ k}\, \hat{P}_k$ ✓

(61) Given $[\hat{L}_i, \hat{V}^j] = i\hbar\, \epsilon_i^{\ j}{}_k\, \hat{V}^k$

5/5

Calculate i) $U_{\vec{\omega}}\, \hat{V}^2\, U_{\omega}^{-1}$

Sol^n: We write $U_{\vec{\omega}}\, \hat{V}^2\, U_{\vec{\omega}}^{-1}$ as

$$U_{\vec{\omega}}\, \hat{V}^2\, U_{\vec{\omega}}^{-1} = U_{\vec{\omega}}\, \hat{V}_i\, \underbrace{U_{\omega}^{-1} U_{\omega}}_{\mathbb{I}}\, \hat{V}^i\, U_{\omega}^{-1}$$ ✓

$\mathbb{I}$ → Inserting identity

$$= U_{\vec{\omega}}\, \hat{V}_i\, U_{\vec{\omega}}^{-1} \cdot U_{\vec{\omega}}\, \hat{V}^i\, U_{\vec{\omega}}^{i}$$ ✓

(61) $U_{\bar{\omega}} \hat{V}^i U_{\bar{\omega}} = \exp\left(\frac{-i}{\hbar} \bar{\omega}.\bar{L}\right) \hat{V}^i \exp\left(\frac{i}{\hbar} \bar{\omega}.\bar{L}\right)$

$$\simeq \left(1 - \frac{i}{\hbar}\bar{\omega}.\bar{L}\right) \hat{V}^i \left(1 + \frac{i}{\hbar}\bar{\omega}.\bar{L}\right)$$

$$\simeq \hat{V}^i - \frac{i}{\hbar}\left[\bar{\omega}.\bar{L}, \hat{V}^i\right]$$ ✓

$$\simeq \hat{V}^i - \frac{i}{\hbar}\omega^j\left[\hat{L}_j, \hat{V}^i\right]$$

$$\simeq \hat{V}^i - \frac{i}{\hbar}\omega^j \, i\hbar \, \epsilon_j{}^i{}_k \hat{V}^k$$

using $[\hat{L}_j, \hat{V}^i] = i\hbar \epsilon_j{}^i{}_k \hat{V}^k$

$$\simeq \hat{V}^i + \omega^j \epsilon_j{}^i{}_k \hat{V}^k \quad \text{— (1)}$$ ok

Now

$$U_{\bar{\omega}} \hat{V}_i U_{\bar{\omega}}^{-1} = \exp\left(\frac{-i}{\hbar}\bar{\omega}.\bar{L}\right) \hat{V}_i \exp\left(\frac{i}{\hbar}\bar{\omega}.\bar{L}\right)$$

$$\simeq \hat{V}_i - \frac{i}{\hbar}\left[\bar{\omega}.\bar{L}, \hat{V}_i\right]$$

$$\simeq \hat{V}_i - \frac{i}{\hbar}\omega_j\left[L^j, \hat{V}_i\right]$$

$$\simeq \hat{V}_i - \frac{i}{\hbar}\omega_j \, i\hbar \, \epsilon^j{}_i{}^k V_k$$

as above

$$\simeq \hat{V}_i + \omega_j \epsilon^j{}_i{}^k \hat{V}_k \quad \text{— (2)}$$

Then $U_\omega \hat{V}^2 U_\omega^{-1} = U_\omega \hat{V}_i U_\omega^{-1} U_\omega V^i U_\omega^{-1}$ becomes

$$= \left(\hat{V}_i + \omega_j \epsilon^{j}{}_{i}{}^{k} \hat{V}_k\right)\left(\hat{V}^i + \omega^j \epsilon_{j}{}^{i}{}_{k} \hat{V}^k\right)$$

$$= \hat{V}_i \hat{V}^i + \underline{\underline{\omega_j \epsilon^{jk}_{i} \hat{V}_k \hat{V}^i}} + \underline{\underline{\omega^j \epsilon_j{}^i{}_k \hat{V}^k \hat{V}_i}} + O(\omega^2)$$

↓ neglect

$$\cong \hat{V}_i \hat{V}^i + 0$$

Note :- The underline terms are identically zero as can be proved by letting $k = i$, and $i = k$ in third term. Note that Levi-Civita is antisymmetric tensor.

Hence

$$U_{\bar{\omega}} \hat{V}^2 U_{\bar{\omega}}^{-1} = \hat{V}^2$$

This proves that ⇒ $\hat{V}^2$ is INVARIANT under ROTATIONAL transformation

→ behaves as scalar under rotation.

Also, it means

→ Commutator of V^2 with $\bar{\omega} \cdot \bar{L}$ is zero.

62. Definition of group SO(3)

3/5 $SO(3) = \{ \text{real } 3\times3 \text{ matrices}; \; RR^T = R^TR = \mathbb{I}, \; \det R = 1 \}$ ✓

We note the Group axioms

① For any pair $a, b \in G$, $c = a * b$ also in in G.

② There exists $e \in G$ $\forall a \in G$ s.t

$$e * a = a * e = a$$

③ $\forall a \in G$, $\exists a^{-1} \in G$ s.t $a^{-1} * a = a^{-1} * a = e$

④ $a * (b * c) = (a * b) * c$ Associativity.

Now, we prove that SO(3) satisfies 1-4

① $R_3 = R_1 R_2$

we have $R_3^T R_3 = (R_1 R_2)^T R_1 R_2$

$$= R_2^T R_1^T R_1 R_2$$

$$= R_2^T R_2 = \mathbb{1}$$ ✓ det R_3 = ?

since $R_2 \in SO(3)$

This proves that even $R_3 \in SO(3)$ (✓)

(2) Now, here identity matrix (3×3) is e

$$R_1 * e = R_1$$

and $R_2 * e = R_2$ ✓.

$e \in SO(3)$?

(3) $\det R_1 = \det R_2 = 1 \neq 0$
↳ non-singular

Since $(R_1^{-1})^T = (R_1^T)^{-1}$

we find $(R_1^{-1})^T R_1^{-1} = (R_1^T)^{-1} R_1^{-1}$

$\det R_1^{-1} = ?$

$$= (R_1^T R_1)^{-1}$$

$$= (\mathbb{1})^{-1} = \mathbb{1} \text{ (✓)}$$

This shows that $R_1^{-1} \in SO(3)$

Therefor, inverse exists and it belongs to the G i.e $SO(3)$

(4) $R_1 \circledast (R_2 \circledast R_3) = (R_1 \circledast R_2) \circledast R_3$

for all square matrices, hence this is satisfies by all $SO(3)$ matrices ✓.

Hence, $SO(3)$ forms a Group ∴

Also, note an additional feature of SO(3).

It is non-abelian i.e $R_1 R_2 \neq R_2 R_1$ ✓

4/4.

(b3) Show that $R_z = \exp(i\omega T_z)$

$$iT_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } R_z = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$i\omega T_z = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$i\omega T_z$ is 3X3

$$e^{i\omega T_z} = (\mathbb{1})_{3\times3} + i\omega T_z + \frac{(i\omega T_z)^2}{2!} + \frac{(i\omega T_z)^3}{3!} + \cdots$$ ✓

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{(i\omega T_z)^2}{2!} + \frac{(i\omega T_z)^3}{3!}$$ ✓

$$(i\omega T_z)^2 = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ✓

$$(i\omega T_z)^3 = \begin{pmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega^3 & 0 \\ -\omega^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ✓

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$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -\omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \frac{1}{3!}\begin{pmatrix} 0 & \omega^3 & 0 \\ -\omega^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots$$

Collecting odd terms and even terms

$$= \begin{pmatrix} 1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} + \cdots & 0 & 0 \\ 0 & 1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\left(\omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} \cdots\right) & 0 \\ \omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega & 0 & 0 \\ 0 & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\sin\omega & 0 \\ \sin\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= R_z$$

(64) $[T_a, T_b] = i C_{ab}{}^c T_c$

Transformation : $T_a = L_a^{\ b} \tilde{T}_b$ - (1)

3/4.

$T_b = L_b^{\ a} \tilde{T}_a$ - (2)

We have $[\tilde{T}_a, \tilde{T}_b] = i \tilde{C}_{ab}{}^c \tilde{T}_c$ - (3)

From (1) & (2) we see

$$\tilde{T}_b = (L^a_{\ b})^{-1} T_a \quad \text{- (3A)}$$

& $$\tilde{T}_a = (L^b_{\ a})^{-1} T_b \quad \text{- (4A)}$$

Possible because only Since L is invertible !!

Using in (3), we get

$$[\tilde{T}_a, \tilde{T}_b] = [(L^a_{\ b})^{-1} T_a, (L^b_{\ a})^{-1} T_b]$$ (✓)

indices!

$$= (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} [T_a, T_b]$$

$$= (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} i C_{ab}{}^c T_c \quad \text{- (5)}$$

Now using (3) & (5)

$$i \tilde{C}_{ab}{}^c \tilde{T}_c = (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} i C_{ab}{}^c T_c$$

$$\tilde{C}_{ab}{}^c \tilde{T}_c = (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} C_{ab}{}^c T_c \quad \text{- (6)}$$

Now, let's look how T_c transforms

$$T_c = L_c^{\ c} \tilde{T}_c$$

using in ⑥, we get

$$\tilde{C}_{ab}^{\ \ c} \tilde{T}_c = (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} C_{ab}^{\ \ c} L_c^{\ c} \tilde{T}_c$$

$$\left(\tilde{C}_{ab}^{\ \ c} - (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} C_{ab}^{\ \ c} L_c^{\ c} \right) \tilde{T}_c = 0$$

Since $\tilde{T}_c$ is arbitrary, we have

$$\tilde{C}_{ab}^{\ \ c} = (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} C_{ab}^{\ \ c} L_c^{\ c}$$

or $\tilde{C}_{ab}^{\ \ f} = (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} C_{ab}^{\ \ c} L_c^{\ f}$ [to avoid confusion of same indices]

✓ ..

ii) $C_{ab}^{\ \ c} = i \epsilon_{ab}^{\ \ c}$

New structure constants for SO(3)

We see that L satisfies

$$L^T L = L L^T = \mathbb{1}$$ why?

i.e $\quad L^T = L^{-1} \Rightarrow (L^a_{\ b})^{-1} = L^b_{\ a}$

Then, we have

$$\tilde{C}_{ab}^{\ \ c} = (L^a_{\ b})^{-1} (L^b_{\ a})^{-1} C_{ab}^{\ \ c} L^c_{\ c}$$

$$= C_{ab}^{\ \ c} L^c_{\ c} \qquad \left[(L^a_{\ b})^{-1} = L^b_{\ a} \right]$$

Therefore,

$$\tilde{C}_{ab}^{\ \ c} = i \epsilon_{ab}^{\ \ c} L_c^{\ c} \quad ✓ ..$$

<u>NOTE</u>: If we see the rotation along z-axis then $L^c_{\ c} = 1$ and we have $\tilde{C}_{ab}^{\ \ c} = i \epsilon_{ab}^{\ \ c}$ ✓.

(65) Representation R of a group G is a homomorphism

2.14. $\quad R: G \to GL(V)$

$$g \mapsto R(g)$$

Homomorphism preserves group structure.

$$R(g_1 \cdot g_2) = R(g_1) \cdot R(g_2)$$ ✓

a) $R(e) = \mathbb{1}$

Let $g_1 = e$ and $g_2 = g$

Then $R(e \cdot g) = R(e) \cdot R(g)$

$$R(g) = R(e) \cdot R(g)$$ ✓

$$0 = R(g)\,[R(e) - \mathbb{1}]$$ Since, our $R(g)$ was arbitrary

no "difference" in G!

$$\Rightarrow \boxed{R(e) = \mathbb{1}}$$ ✓

b) Prove $R(g^{-1}) = R(g)^{-1}$

~~Now let's take $g_1 = g^{-1}$ and $g_2 = e$~~

~~$$R(g_1 \cdot g_2) = R(g_1) \cdot R(g_2)$$~~

~~$$R(g^{-1} \cdot e) = R(e)\, R(g)^{-1}$$~~

~~$$R(g^{-1}) = R(g)^{-1}$$~~

65.b $\forall g \in G \ \exists \ g^{-1} \in G$ s.t

$$g * g^{-1} = e$$

$$R(g * g^{-1}) = R(e) = R(g) * R(g^{-1})$$ ✓

But from 65.a $R(e) = 1$

$$R(e) = R(g) * R(g^{-1})$$

$$\Rightarrow \quad R(g^{-1}) = \frac{1}{R(g)}$$ ✓

$$= R(g)^{-1}$$ ✓.

66. Given :-

4/6. $[T_a, T_b] = i\, C_{ab}^{c}\, T_c$

and elements $t_a \in gl(V)$ with

$$[t_a, t_b] = i\, C_{ab}^{c}\, t_c$$

R. Jha
11/07/2013
Quantum Assignment

8/10

The linear map $\Rightarrow r(T_a) = t_a$ is a '<u>rep</u>' of Lie(G).

** $$r([T_a, T_b]) = r(i\, C_{ab}^{c}\, T_c)$$

$$= i\, C_{ab}^{c}\, r(T_c) \checkmark$$

Linear map here
$\Rightarrow r(T_a) = t_a$

gives, $$r([T_a, T_b]) = i\, C_{ab}^{c}\, t_c$$

$$= [t_a, t_b]$$

arbitrary elements in Lie(G) ?

But, the map $r(T_a) = t_a$ implies

$$= [r(T_a), r(T_b)]$$

We have $$r([T_a, T_b]) = [r(T_a), r(T_b)] \checkmark$$

Hence, it furnishes a representation of Lie(G) . .

67. Recall from the Lecture 19 notes,

4/4. $$\langle j\, m_a | J_x | j\, m_b \rangle = \frac{\hbar}{2}\left[\delta_{m_a, m_b+1} + \delta_{m_a, m_b-1}\right]\sqrt{j(j+1) - m_a m_b} \quad -(1)$$

$$\langle j\, M_a | J_y | j\, m_b \rangle = \frac{\hbar}{2}\left[\delta_{m_a, m_b+1} - \delta_{m_a}, \delta_{m_b-1}\right]\sqrt{j(j+1) - m_a m_b} \quad -(2)$$

and $\langle j\, m_a | J_z | j\, m_b \rangle = \hbar\, \delta_{m_a, m_b}\, m_a$ — ③

$$J_x = \begin{pmatrix} \langle \frac{1}{2}\ \frac{1}{2} | J_x | \frac{1}{2}\ \frac{1}{2} \rangle & \langle \frac{1}{2}\ \frac{1}{2} | J_x | \frac{1}{2}\ \frac{-1}{2} \rangle \\ \langle \frac{1}{2}\ \frac{-1}{2} | J_x | \frac{1}{2}\ \frac{1}{2} \rangle & \langle \frac{1}{2}\ \frac{-1}{2} | J_x | \frac{1}{2}\ \frac{-1}{2} \rangle \end{pmatrix}$$

Matrix rep. for J_x
↳ for $j = \frac{1}{2}$

using ① we get

$$= \begin{pmatrix} 0 & \frac{\hbar}{2}\sqrt{j(j+1) - \frac{1}{2}(\frac{-1}{2})} \\ \frac{\hbar}{2}\sqrt{\frac{1}{2}\times\frac{3}{2} + \frac{1}{4}} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$J_y = \begin{pmatrix} \langle \frac{1}{2}\ \frac{1}{2} | J_y | \frac{1}{2}\ \frac{1}{2} \rangle & \langle \frac{1}{2}\ \frac{1}{2} | J_y | \frac{1}{2}\ \frac{-1}{2} \rangle \\ \langle \frac{1}{2}\ \frac{-1}{2} | J_y | \frac{1}{2}\ \frac{1}{2} \rangle & \langle \frac{1}{2}\ \frac{-1}{2} | J_y | \frac{1}{2}\ \frac{-1}{2} \rangle \end{pmatrix}$$

using ② we get

$$= \begin{pmatrix} 0 & \frac{\hbar}{2i} \\ -\frac{\hbar}{2i} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_z = \begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix} = \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(under the $\hbar$ in the top-left entry: [redacted])

These matrices are nothing but the Pauli-spin matrices. σ_x, σ_y & σ_z. They occur here because $j = 1/2$ means that there is only '$\underline{s}$' (spin ang. momentum) contribution here and no $\underline{\ell}$ (orb. ang momentum) !!

Pauli matrices are traceless & anti-commute as it is evident in the J_x, J_y, J_z we got here.

$$D^j_{mm'} = \langle jm | U_\omega | jm' \rangle$$

$|\psi\rangle \in \mathcal{H} \in C^2$

$R \in SO(3)$

↓ divide & rule

Infinitesimal ..

rotation don't correspond to SO(3) but to tangent space of it.

↓

Lie algebra of so(3)

↓ acts on tangent space.

Raghav jha
Govind jha
dd...

1) Lie group rep
2) Group rep
✓ 3) Spin op
✓ 4) Sim Conn
✓ 5) Projective Rep
✓ 6) Y_{lm} → H atom
✓ 7) Spin ½ rotation
✓ 8) Wigner
✓ 9) SU(2)
10)

$$U_{2\pi} |jm\rangle = (-1)^{2j} |jm\rangle$$

if $j = 1/2$

$$U_{2\pi} |jm\rangle = (-1) |jm\rangle$$

$$U_{4\pi} |jm\rangle = |jm\rangle$$

$\mathbb{1} \in SU(2)$ AND $-\mathbb{1} \in SU(2)$

⇓ Double cover.

$\mathbb{1} \in SO(3)$

May be God may have cadmelted it Mamism. for the moment we consider theoretically

12.12.2011

for $j = 3/2$

$$U_{2\pi} |jm\rangle = -1 \, |jm\rangle$$

$$e^{\frac{-i}{\hbar} H \Delta t} = e\left(\frac{i}{\hbar} g \frac{e}{2mc} \vec{S} \cdot \vec{B} \Delta t\right).$$

68. $J_z|\psi\rangle = m\hbar|\psi\rangle$ (20/24) K. JHA

4/4.

Let us recall Wigner function given by:

$$\mathcal{D}^{j}_{m'm}(R) = \left\langle j\, m' \middle| \exp\left(-\frac{i\hat{J}.n\phi}{\hbar}\right) \middle| j\, m \right\rangle$$

too complicated in this context.

Rotation in general cannot change 'j' value but it does change 'm' value

The Rotation operator (Wigner Rotation operator) is given by for rotation around z-axis

$$\hat{R}(\phi\hat{k}) = e^{-\frac{i\hat{J}_z\phi}{\hbar}}$$ ✓

Now we rotate state $|\psi\rangle$ by δ around z-axis

$$\hat{R}(\phi\hat{k})|\psi\rangle = \left[1 - \frac{i\phi\hat{J}_z}{\hbar} + \frac{1}{2!}\left(-\frac{i\phi\hat{J}_z}{\hbar}\right)^2 + \cdots\right]|\psi\rangle$$

$\hat{J}_z|\psi\rangle = m\hbar|\psi\rangle$ [here $\delta = \phi$]

$\hat{J}_z^2|\psi\rangle = m^2\hbar^2|\psi\rangle$

and so.. on

$$\hat{R}(\phi\hat{k})|\psi\rangle = \left[1 - i\phi m + \frac{1}{2!}(-i\phi m)^2 + \dots\right]|\psi\rangle$$

$$= e^{-i\phi m}|\psi\rangle \checkmark$$

(from previous Exercise)

The state of the system after rotation is

$$\boxed{e^{-i\phi m}|\psi\rangle}$$

ii) $|\psi'\rangle = e^{-i\phi m}|\psi\rangle$

Now state vector remain invariant if $\underline{e^{-i\phi m} = 1}$ ✓ $\Rightarrow \phi m = 2\pi$

$$\boxed{\phi = \frac{2\pi}{m} \cdot n}$$

↳ except for $m = 0$

m odd $\underset{\text{INVARIANT}}{\Longrightarrow}$ $\phi = \frac{2\pi}{1}, \frac{2\pi}{3} \dots$ ✓

m even $\underset{\text{INVARIANT}}{\Longrightarrow}$ $\phi =$ [redacted] π, [redacted] $\frac{\pi}{2}, \dots$ ✓

70. $f(\theta,\phi) = \sum_{l,m} f_{lm} Y_{lm}(\theta,\phi)$

4/4

Let's multiply by $Y^*_{l'm'}(\theta,\phi)$ on both sides and integrate over the solid angle Ω. ✓

$$\int f(\theta,\phi) Y^*_{l'm'}(\theta,\phi)\, d^2\Omega = \int \sum f_{lm} Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi)\, d^2\Omega \quad - ①$$

Now, from lecture notes we know

$$\int d^2\Omega\, Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{l'l}\,\delta_{m'm} \quad - ②$$

Using ② in ① gives

$$f_{lm} = \int f(\theta,\phi) Y^*_{lm}(\theta,\phi)\, d^2\Omega$$ ✓

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, f(\theta,\phi) Y^*_{lm}(\theta,\phi)$$ ✓

b) We have

~~$$f_{lm} = \int f(\theta,\phi) Y^*_{lm}\, d^2\Omega \quad - (2A)$$~~

10.6. We have $f_{lm} = \int f(\theta,\phi)\, Y^*_{lm}(\theta,\phi)\, d^2\Omega$

Take complex conjugate on both sides and use the fact that $f(\theta,\phi) = f^*(\theta,\phi)$, we get

$$f^*_{lm} = \int f(\theta,\phi)\, Y_{lm}(\theta,\phi)\, d^2\Omega \quad - (1)$$

Now use the result for spherical harmonics that

$$Y^*_{lm}(\theta,\phi) = (-1)^m Y_{l-m}(\theta,\phi) \quad - (2)$$

$$\Rightarrow \quad Y_{lm}(\theta,\phi) = (-1)^m Y^*_{l-m}(\theta,\phi) \quad - (3)$$

Use (3) in (1) we get

$$f^*_{lm} = \int f(\theta,\phi)\, (-1)^m\, Y^*_{l-m}(\theta,\phi)\, d^2\Omega$$

$$= (-1)^m \int f(\theta,\phi)\, Y^*_{l-m}(\theta,\phi)\, d^2\Omega$$

$$= (-1)^m f_{l-m}$$

Again take complex conjugate

$$f_{lm} = (-1)^m f^*_{l-m}$$

71. $J^2|\psi\rangle = 2\hbar^2|\psi\rangle$

7/8. Since we know $J^2|\psi\rangle = j(j+1)\hbar^2|\psi\rangle$

$$\Rightarrow \boxed{j = 1} \checkmark$$

Now, we also have $J_x|\psi\rangle = \hbar|\psi\rangle$

Eigenvectors of J_z are given by:

$|1\ 1\rangle$, $|1\ 0\rangle$ and $|1\ -1\rangle$ ✓

Let $|\psi\rangle$ → EIGENVECTOR OF J_x

$$|\psi\rangle = a|1\ -1\rangle + b|1\ 0\rangle + c|1\ 1\rangle \checkmark$$

now

$$J_x|\psi\rangle = \hbar\left[a|1\ -1\rangle + b|1\ 0\rangle + c|1\ 1\rangle\right] \quad - (1)$$

We also know that

$$J_+ = J_x + iJ_y \checkmark$$
$$J_- = J_x - iJ_y$$

$$\Rightarrow \quad J_x = \frac{1}{2}(J_+ + J_-)$$

$$J_x|\psi\rangle = \frac{1}{2}(J_+ + J_-)|\psi\rangle \checkmark$$

$$= \frac{1}{2} J_+ |\psi\rangle + \frac{1}{2} J_- |\psi\rangle$$

$$= \frac{1}{2} J_+ \left[a|1\,-1\rangle + b|1\,0\rangle + c|1\,-1\rangle \right] + \frac{1}{2} J_- \left[a|1\,-1\rangle + b|1\,0\rangle + c|1\,1\rangle \right]$$

$$= \frac{a}{2}\sqrt{2}\,|1\,0\rangle + \frac{b}{2}\sqrt{2}\,|1\,1\rangle + \frac{b}{2}\sqrt{2}\,|1\,-1\rangle + \frac{c}{2}\sqrt{2}\,|1\,0\rangle$$

⇓

[we used
$J_+|1\,1\rangle = 0$
$J_+|1\,0\rangle = \sqrt{2}\,|1\,1\rangle$
$J_+|1\,-1\rangle = \sqrt{2}\,|1\,0\rangle$
$J_-|1\,1\rangle = \sqrt{2}\,|1\,0\rangle$
$J_-|1\,0\rangle = \sqrt{2}\,|1\,-1\rangle$
$J_-|1\,-1\rangle = 0$

$$= \frac{a+c}{\sqrt{2}}\,|1\,0\rangle + \frac{b}{\sqrt{2}}\,|1\,1\rangle + \frac{b}{\sqrt{2}}\,|1\,-1\rangle$$

but $a^2+b^2+c^2 = 1$, since $|\psi\rangle$ as a linear combination should be normalized

$$\Rightarrow \quad a = c, \quad b = \sqrt{2}\,a \quad \Rightarrow \quad a = 1/2 = c, \quad b = 1/\sqrt{2}$$

Hence,

$$|\psi\rangle = \frac{1}{2}|1\,-1\rangle + \frac{1}{\sqrt{2}}|1\,0\rangle + \frac{1}{2}|1\,-1\rangle$$

b. Rotation operator acting on $|1\ 1\rangle$ by $90°$ along y-axis.

* If we parametrize U_ω by the three Euler angles of rotation, $U_\omega = U_\omega(\varphi, \theta, \psi)$

$$\mathcal{D}^{l}_{m'm} = e^{-i\varphi m'} d^{l}_{mm'}(\theta)\, e^{-i\psi m}$$ ✓

Since we rotate around y-axis, we have

$$\mathcal{D}^{l}_{m'm} = d^{l}_{mm'}(\theta)$$ ✓

where $d^{l}_{mm'}(\theta) = \langle jm | e^{-i\theta L_y} | jm' \rangle$

Here, we have to rotate $|1\ 1\rangle$ which can be expressed as $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ✓.

$$|1\ 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

SK

$$|1\ 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|1\ {-1}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

m can have 3 possible values i.e -1, 0, 1

We look up Wikipedia for $d^{l}_{mm'}$ for j = 1 and then act on $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$= \begin{pmatrix} 1/2 & -1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix} \begin{pmatrix}1\\0\\0\end{pmatrix}$$ OK.

$$= \begin{pmatrix}1/2\\1/\sqrt{2}\\1/2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1/\sqrt{2}\\1\\1/\sqrt{2}\end{pmatrix}$$ ✓

$$= \frac{1}{2}\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$= |\psi\rangle$ ✓ as found in part (A)

Yes, they agree. They are the same. ✓

c) This happens because for $l = 1$, m can have three possible values i.e $m = -1, 0, 1$

If, we rotate along the y-axis, we are basically affecting x & z axis.

$l = 1$ <u>REMINDS</u> OF atomic orbitals ($\underline{p}$)

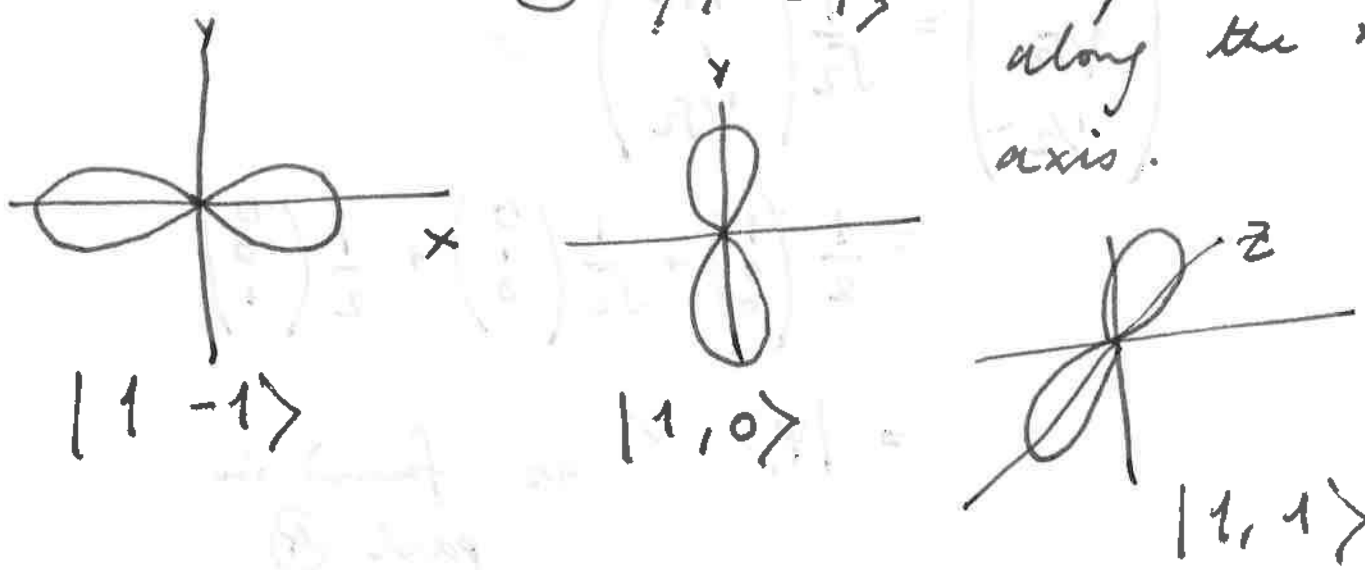
	l value
s	0
p	1
d	2
f	3

Now

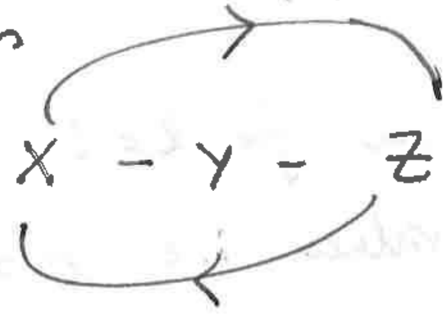
$|1\ 1\rangle$

$|1\ 0\rangle$ can be represented along the x-y & z axis.

$|1\ -1\rangle$



Rotating $|1,1\rangle$ about y-axis means



Rotation

X → Z

Z → X

Y → Y

Now, since the state $|1\ 1\rangle$ and $|1\ -1\rangle$ also have the same factor in front. Basically nothing changes.

Hence, both the answers to part a & part b are the same !! ?

Rotating eigenvector of J_z i.e $|1\ 1\rangle$ around y-axis gives eigenvector of J_x i.e $|\psi\rangle$

72. $\mathcal{R} : \psi(\vec{x}) \Rightarrow \psi'(\vec{x}) = D^{(j)}(R)\,\psi(R\vec{x})$ OK.

1/4. Show that this map is not homomorphism.

Homomorphism means for a map ϕ

$$\phi(g_1 \cdot g_2) = \phi(g_1)\cdot\phi(g_2) \quad \forall\ g_1, g_2 \in G$$ ✓

We'll do it in two ways

1) Rotate by ~~R_1R_2~~ R_2R_1 at once

2) Rotate by R_1 and then rotate by R_2.

$$\psi'(\bar{x}) = \mathcal{D}^{(j)}(R_1)\,\psi(R_1\vec{x}) \quad (\text{rotating by } R_1)$$

Now we rotate $\psi'(\bar{x})$ by R_2 and get

$$\psi''(\bar{x}) = \mathcal{D}^{j}(R_2)\,\mathcal{D}^{j}(R_1)\,\psi(R_2R_1\vec{x}) \quad \times \quad — \quad (1)$$

SECOND WAY: Rotate by R_1R_2 (R_2R_1) at once

$$\psi''(\bar{x}) = \mathcal{D}^{j}(R_1R_2)\,\psi(R_1R_2\vec{x}) \quad — \quad (2)$$

Now, since $\mathcal{D}^{(j)}(R)$ is a representation of $SO(3)$,

we have $\mathcal{D}^{(j)}(R_1R_2) = \mathcal{D}^{j}(R_2)\,\mathcal{D}^{j}(R_1)$ $\times$

↳ Group property !!

But we note that $R_2R_1 \neq R_1R_2$ (rotation matrices don't commute in general)...

Hence

$\psi''(\vec{x})$ from (1) and (2) are not EQUAL. $\times$

i.e. $\psi(R_1R_2\vec{x}) \neq \psi(R_1\vec{x}) \cdot \psi(R_2\bar{x})$

$R: \psi(\bar{x}) \to \psi'(\bar{x}) = \mathcal{D}^{j}(R)\psi(R\bar{x}) \rightarrow$ NOT A HOMOMORPHISM !

Now, we see why the correct transformation

is $\psi'(\bar{x}) = \mathcal{D}^{j}(R)\,\psi(R^{-1}\bar{x})$

Which is A homomorphism

Here we get $\psi''(\bar{x}) = \mathcal{D}^{j}(R_1 R_2)\,\psi((R_1 R_2)^{-1}\bar{x})$ * (above: $R_2 R_1$, $R_2 R_1$)

and $\psi''(\bar{x}) = \mathcal{D}^{j}(R_2)\,\mathcal{D}^{j}(R_1)\,\psi(R_2^{-1} R_1^{-1}\vec{x})$ *

They match and are equal because

$$(R_1 R_2)^{-1} = R_2^{-1} R_1^{-1}$$

since R_1 & R_2 are both orthogonal matrices.

and hence

HOMOMORPHISM

~~$\psi(R_1 R_2 \bar{x}) = \psi(R_1 \bar{x}) \cdot \psi(R_2 \bar{x})$~~

$R: \ \psi(\bar{x}) \rightarrow \psi'(\bar{x}) = \mathcal{D}^{j}(R)\,\psi(R^{-1}\vec{x})$

is a HOMOMORPHISM.

THANK YOU FOR THE HINT !

69. Prove $Y_{lm}(\theta, \varphi) \overset{?}{=} (-1)^l Y_{lm}(\theta, \varphi)$

4.4.

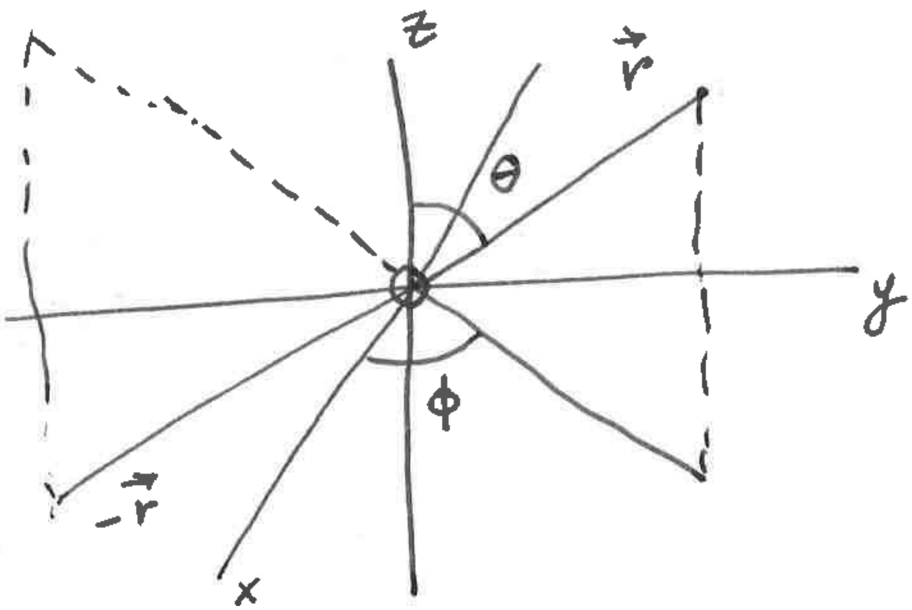
We have

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_{lm}(\cos\theta) e^{im\varphi} \quad \text{— (A)}$$

and $$P_{lm}(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

Under parity transformation, we note that

$$(r, \theta, \phi) \longrightarrow (r, \pi-\theta, \phi+\pi)$$



$$\cos\theta \longrightarrow \cos(\pi-\theta) = -\cos\theta$$

Now $$P_{lm}(\cos\theta) \xrightarrow{\text{PARITY}} (-1)^{l+m} P_{lm}(\cos\theta) \quad \text{— (1)}$$

$$e^{im\phi} \xrightarrow{\text{PARITY}} e^{im(\phi+\pi)}$$

$$(-1)^m e^{im\phi} \checkmark$$

(ς in the given question is now $\underline{\underline{\phi}}$)

or, in our notation

$$e^{im\varsigma} \xrightarrow{\text{PARITY}} (-1)^m e^{im\varsigma} \quad - \text{②}$$

Using ① & ② in Ⓐ gives

$$Y_{lm}(\theta,\varsigma) \xrightarrow{\text{PARITY}} (-1)^{l+m}(-1)^m$$

$$= (-1)^l (-1)^{2m}$$

$$\rightarrow (-1)^l \checkmark$$

Since m = positive or negative integer

$2m$ = always ⊕ve integer

$(-1)^{2m} = 1$

Hence, we prove

$$Y_{lm}(\theta,\varsigma) \longrightarrow (-1)^l Y_{lm}(\theta,\varsigma) \checkmark$$

73. Show that R(A) defined by $(R(A)\vec{x})\cdot\vec{\sigma} = A\vec{x}\cdot\vec{\sigma}A^{-1}$ is a homomorphism from SU(2) into SO(3)

R. JHA
11/21/2013

14/24

Solution: We have to show that $R(A_1A_2) = R(A_1)R(A_2)$ for $A_1, A_2 \in SU(2)$

2/6. and $R(A) \in SO(3)$!

$\in SU(2)$

$R(A_1A_2)\vec{x}\ (R(A_1A_2)\vec{x})\cdot\vec{\sigma}$ — $\in SO(3)$

$= A_1A_2\,\vec{x}\cdot\vec{\sigma}\,(A_1A_2)^{-1} := A_1A_2\,\vec{x}\,\vec{\sigma}\,A_2^{-1}A_1^{-1}$

$= A_1\,(A_2\,\vec{x}\cdot\vec{\sigma}\,A_2^{-1})\,A_1^{-1}$

$R(A_2)\vec{x}\cdot\vec{\sigma}$

$= A_1\,R(A_2)\,A_1^{-1}$

$= (R(A_1).R(A_2)\,\vec{x})\cdot\vec{\sigma}$

Since $A_1, A_2 \in SU(2)$ we have $(A_1A_2)^{+}(A_1A_2) = \mathbb{1}$
$\Rightarrow (A_1A_2)^{-1} = (A_1A_2)^{+} = A_2^{+}A_1^{+} = A_2^{-1}A_1^{-1}$

1) Stat Mech
2) EMT
3) QFT-II

Pure States
– Quantum Information
– mixed – pure

Hence, the map $\mapsto R(A)$ is a homomorphism. ok

It's a 2-1 homomorphism.

If we form a coset $\frac{SU(2)}{Z2}$, then we are removing half the elements of SU(2) and we have ISOMORPHISM. ok

$$\frac{SU(2)}{Z/2} \xrightarrow{\text{Isomorphism}} SO(3)$$

Also $SO(3,1) \rightarrow$ Special Lorentz Group

$$SO(3,1) \approx \frac{SU(2)\otimes SU(2)}{Z2}$$

SO(3) is doubly covered by SU(2).

74. Suppose 'S' is superselected observable

3/6. and $|\psi\rangle = \alpha|s_1\rangle + \beta|s_2\rangle$ when $\hat{S}|s_1\rangle = s_1|s_1\rangle$, $\hat{S}|s_2\rangle = s_2|s_2\rangle$

Show $\mathcal{S} = |\alpha|^2|s_1\rangle\langle s_1| + |\beta|^2|s_2\rangle\langle s_2|$ has same expection for A as $\langle\psi|\hat{A}|\psi\rangle$

Let's calculate $\langle\psi|\hat{A}|\psi\rangle$ first, $|\psi\rangle = \alpha|S_1\rangle + \beta|S_2\rangle$

$\langle\psi| = \langle S_1|\alpha^* + \langle S_2|\beta^*$

So, $\langle\psi|\hat{A}|\psi\rangle = ?$ (Need to find out)

Two ways to do this.

① spectral decomposition of $A = \sum a|a\rangle\langle a|$

② since $[S, A] = 0$, eigenvector of S, are also eigenvectors of A.

So, we can, have (without any loss of generality)

$$\hat{A}|S_1\rangle = a_1|S_1\rangle \text{ and } \hat{A}|S_2\rangle = a_2|S_2\rangle$$ ok

also $|S_1\rangle$ and $|S_2\rangle$ are ORTHOGONAL to each other ✓

so $\langle S_1|S_2\rangle = 0$

i.e CROSS TERMS CANCEL

So, we get

$$\langle\psi|\hat{A}|\psi\rangle = |\alpha|^2 a_1 + |\beta|^2 a_2$$ ✓ ①

Now, Since $\rho = |\alpha|^2|S_1\rangle\langle S_1| + |\beta|^2|S_2\rangle\langle S_2|$

We know that expectation value of any observable will be equal to Trace(ρA)..

Let's calculate $\langle\xi''|\rho|\xi'\rangle$ [actually elements of corresponding density matrix)

$$\langle\xi''|\rho|\xi'\rangle = |\alpha|^2\langle\xi''|S_1\rangle\langle S_1|\xi'\rangle + |\beta|^2\langle\xi''|S_2\rangle\langle S_2|\xi'\rangle$$ ok

Now, we find

$$\langle\xi'|A|\xi''\rangle = \langle\xi'|\sum_{1,2} a|a\rangle\langle a|\xi''\rangle$$ ✗

[can be also generalized to involve till N]

$$= a_1\langle\xi'|a_1\rangle\langle a_1|\xi''\rangle + a_2\langle\xi''|a_2\rangle\langle a_2|\xi'\rangle$$ ✗

$$\text{Trace}(\rho A) = \sum_{\xi'\xi''}\langle\xi''|\rho|\xi'\rangle\langle\xi'|A|\xi''\rangle$$ } ?

$$= |\alpha|^2 a_1\langle S_1|a_1\rangle\langle a_1|S_1\rangle + |\beta|^2 a_2\langle S_2|a_2\rangle\langle a_2|S_2\rangle$$

$$= |\alpha|^2 a_1 + |\beta|^2 a_2$$ Since $\langle a_1|S_1\rangle = 1$, $\langle a_2|S_2\rangle = 1$

$\text{Trace}(\rho A) = |\alpha|^2 a_1 + |\beta|^2 a_2$

$= \langle \psi | \hat{A} | \psi \rangle$ ✓

Both are equal.

$\langle a_1 | s_1 \rangle = 1$
$\langle a_2 | s_2 \rangle = 1$
$\langle a_1 | a_2 \rangle = 0$
$\langle s_1 | s_2 \rangle = 0$

———— x ————

75. We place a particle in a constant magnetic field $\vec{B}$. Calculate $\langle \vec{S} \rangle (t)$

Solⁿ: This problem can be solved by considering the magnetic field along z direction. OK.

4/6.

Then, $\langle S_z \rangle$ will not change with time but $\langle S_x \rangle$ and $\langle S_y \rangle$ will.

ALSO, we might get slightly different expressions depending on initial state we choose..

$\vec{B} = B_0 \hat{k}$ OK $\Delta \hat{H} = -\vec{m} \cdot \vec{B}$

$= -g \frac{e}{2mc} \vec{S} \cdot \vec{B}$ ✓

So, it means S_z eigenstates are also energy eigenstates

$E_\pm = \mp \frac{ge}{2mc} \frac{\hbar}{2} B_0$

Let $\omega_\mp = \frac{ge}{2mc} B_0$; then $H = \omega S_z$

Time evolution operator reads,

$U(t,0) = \exp\left(\frac{-i\omega S_z t}{\hbar}\right)$ ✓

We expand the initial state in terms of $|+\rangle$ and $|-\rangle$ [eigenkets of S_z and $\hat{H}$] why spin 1/2 ?

at $t = 0$

$$|\alpha\rangle = c_+|+\rangle + c_-|-\rangle$$

$$|\alpha(t)\rangle = C_+ \exp\left(\frac{-i\omega t}{2}\right)|+\rangle + C_- \exp\left(\frac{+i\omega t}{2}\right)|-\rangle$$ ✓

[we know
$H|\pm\rangle = \frac{\pm\hbar\omega}{2}|\pm\rangle$
since
$\left[S_{z_\pm} = \pm\frac{\hbar}{2}\right]$]

LET US SUPPOSE NOW THAT INITIALLY PARTICLE WAS IN $S_x +$ state.

$$|\alpha\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

OK, a bit unintuitive.

Probability to find it in $S_x \pm$ state is

$$|\langle S_x \pm; t | S_x +, 0\rangle|^2 = \begin{cases} \cos^2\frac{\omega t}{2} & \text{for } S_x + \\ \sin^2\frac{\omega t}{2} & \text{for } S_x - \end{cases}$$

So, we started off with spin in positive x-direction but due to magnetic field, things changed.

$$\langle S_x\rangle(t) = \frac{\hbar}{2}\cos^2\left(\frac{\omega t}{2}\right) - \frac{\hbar}{2}\sin^2\left(\frac{\omega t}{2}\right)$$

$$= \frac{\hbar}{2}\cos\omega t$$ ✓

Similarly

$$\langle S_y\rangle(t) = \frac{\hbar}{2}\sin\omega t$$ ✓

and $\langle S_z\rangle(t) = 0$ → Since this does not change. Initially → 0
Final → 0.
Since $[B, S_z] = 0$

This means spin precesses in the x-y plane.

ANOTHER METHOD :- Suppose we don't want to confine initial state to ⊕ve-x direction. better.

we can define

$$|\psi(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$ where let's take
$a = \cos\alpha/2$
$b = \sin\alpha/2$ modulo phases.
i.e $a^2 + b^2 = 1$

Now, $$|\psi(t)\rangle = \begin{pmatrix} \cos\alpha/2 \, e^{\frac{i\omega_0 t}{2}} \\ \sin\alpha/2 \, e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}$$ ✓ where $\omega = \frac{ge}{2mc} B_0$

Let's calculate ..

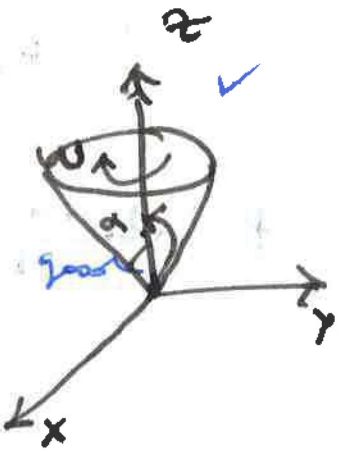
$$\langle S_x\rangle(t) = \langle\psi(t)|S_x|\psi(t)\rangle$$
$$= \left(\cos\alpha/2 \, e^{-\frac{i\omega_0 t}{2}} \quad \sin\alpha/2 \, e^{\frac{i\omega_0 t}{2}}\right) \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha/2 \, e^{\frac{i\omega_0 t}{2}} \\ \sin\alpha/2 \, e^{-\frac{i\omega_0 t}{2}} \end{pmatrix}$$

$$= \hbar/2 \sin\alpha \cos\omega_0 t$$

Similarly

$$\langle S_y\rangle(t) = -\frac{\hbar}{2}\sin\alpha \sin(\omega_0 t)$$

and $\langle S_z\rangle(t) = \frac{\hbar}{2}\cos\alpha$ (independent of t) ✓
↓ imp. point

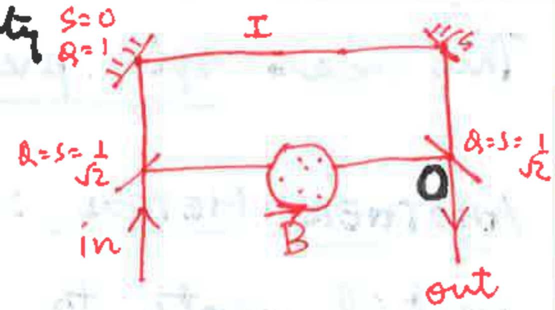


FINALLY,

$\langle\vec{S}\rangle(t)$ ~~=~~ $\langle S_x\rangle(t) + \langle S_y\rangle(t) + \langle S_z\rangle(t)$ (scalars)
vector
which can be readily calculated.

76. This problem gives the same probability with two different starting state. (initial state)

5/6.



Since, the neutrons are polarized, it means that the direction has been uniquely fixed.

Lets start with $|+z\rangle$ ✓

Region I → No change → $\frac{1}{\sqrt{2}}|+z\rangle$ ✓

But the $\frac{1}{\sqrt{2}}|+z\rangle$ that goes to $\vec{B}$ faces a rotation.

using Problem (75), we denote the transformation operation as $e^{-\frac{i\omega t}{2}}$. ✓

Hence we get the state $\frac{1}{\sqrt{2}} e^{-\frac{i\omega t}{2}}|+z\rangle$ after passing through $\vec{B}$. ✓

They now meet at O (see figure above) and the out state becomes

$$|out\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}} e^{-\frac{i\omega t}{2}}|+z\rangle + \frac{1}{\sqrt{2}}|+z\rangle\right]$$ ✓

$|in\rangle = |+z\rangle$

why this?

Probability of detecting neutrons $= |\langle \psi_{out}|\psi_{in}\rangle|^2$ (✓)

$$= \frac{1}{2}\left[1 + \cos\left(\frac{\omega t}{2}\right)\right]$$ ✓

$|\langle \psi_{out}|\psi_{out}\rangle|^2$

1. Potential terms (P.T) 75/80 R. JHA

01/28/13

$$i\frac{\partial}{\partial t} f(n_1 \ldots n_\infty; t) = \sum_i \langle i|T|i\rangle n_i f(n_1 \ldots n_\infty; t)$$

$$+ \sum_{i \neq j} \langle i|T|j\rangle \sqrt{n_i}\sqrt{n_j+1}\, f(n_1 \ldots n_{i-1} \ldots n_{j+1}, n_\infty; t)$$

$$+ \text{P.T.} \quad \ldots (1)$$

For P.T, we get from notes

$$\text{P.T} = \frac{1}{2}\sum_{k\neq l}\sum_{Q,Q'} \int d\bar{x}_k \int d\bar{x}_l\, \psi^\dagger_{q_k}(\bar{x}_k)\psi_{q_l}(\bar{x}_l) V(x_k, x_l)\, \psi_Q(\bar{x}_k)\psi'_{Q}(\bar{x}_l) \times B$$

$$\text{where } B = C(q_1 \ldots q_{k-1}, Q, q_{k+1} \ldots Q', q_{l+1} \ldots q_N; t) \ldots (2)$$

The integral can be written simplified as.

using $|q_k\rangle = |i\rangle$ and $|q_l\rangle = |j\rangle$ in .... (3)

$|Q\rangle = |k\rangle$ and $|Q'\rangle = |l\rangle$

Integrand of

$$\text{P.T} = \int d\bar{x}_k \int d\bar{x}_l \langle q_k|\bar{x}_k\rangle\langle q_l|\bar{x}_l\rangle V(\bar{x}_k, \bar{x}_l)\langle \bar{x}_k|k\rangle\langle \bar{x}_l|l\rangle$$

$$= \langle i|\langle j| \int d\bar{x}_k \underbrace{\int d\bar{x}_l\, |\bar{x}_k\rangle\langle \bar{x}_k|}_{\mathbb{1}} V(\bar{x}_k, \bar{x}_l) \underbrace{\int d\bar{x}_l |\bar{x}_l\rangle\langle \bar{x}_l|}_{\mathbb{1}}\ |k\rangle|l\rangle$$

$\langle i|\langle j| \equiv \langle ij|$

$|k\rangle|l\rangle = |kl\rangle$

$$= \langle ij|V|kl\rangle \quad ✓$$

Next important step is to rewrite the expression for C from eq. (2)

We can also guess that potential term involves two creation operation & two annihilation.

$$C(q_1 \dots q_{k-1}, Q, q_{k+1} \cdots q_{l-1}, Q' q_{l+1} \cdots q_N; t)$$

$$\xrightarrow{\text{replaced}} \bar{C}(n_1 \dots n_i - 1 \dots n_j - 1, \dots n_k + 1 \dots n_l + 1 \dots; t)$$

$$P.T = \frac{1}{2} \sum_{\substack{i \neq k \\ j \neq l}} \langle ij | V | kl \rangle f(n_1 \dots n_i - 1, n_j - 1; \dots n_k + 1 \dots n_l + 1) \sqrt{n_i (n_k + 1) n_j (n_l + 1)}$$

if $i = k$ then we add this term, which is

$$\frac{1}{2} \sum_{j \neq l} \langle ij | V | kl \rangle \; n_i \sqrt{n_j (n_l + 1)} f(n_1 \dots n_i \dots n_j - 1 \dots n_l + 1 \dots, t$$

In case of $j = l$

$$= \frac{1}{2} \sum_{i \neq k} \langle ij | V | kl \rangle \; n_j \sqrt{n_i (n_k + 1)} f(- - - ; t) \quad - \text{②}$$

many other cases as well!

$$P.T = \frac{1}{2} \sum_{\substack{i \neq k \\ j \neq l}} \langle ij | V | kl \rangle f(n_1 \dots n_i - 1 \dots n_j - 1 \dots n_k + 1 \dots n_k + 1; t)$$

$$+ \frac{1}{2} \sum_{k \neq l} \langle ij | V | kl \rangle f(n_1 \dots n_i - 1 \dots n_k + 1$$

$$+ \dots$$

from ②

rep. in terms of b's? Do coefficients match?

2. $N = \int d^3x \; \phi^\dagger(x)\, \phi(x)$

$\hat{T} = \int d^3x \; \phi^\dagger(x)\, T(x)\, \phi(x)$; $\hat{V} = \frac{1}{2}\int d^3x\, d^3y \; \phi^\dagger(x)\, \phi^\dagger(y)\, V(x,y)\, \phi(x)\, \phi(y)$

Let's prove $[N, \phi(x)]$ first.

$$[\hat{N}, \phi(x)] = \int d^3x' \; \phi^\dagger(x') \, [\phi(x'), \phi(x)] + \int d^3x' \, [\phi^\dagger(x'), \phi(x)]\phi(x)$$

$$= -\phi(x) \quad \cdots \quad (1)$$

↳ Also proved in Q.4.

$$[\hat{N}, \phi^\dagger(x)] = \phi^\dagger(x) \quad \cdots \quad (2)$$

Now, we use a commutator identity which involves shifting through 'n' places.

$$[A, B_n \ldots B_1] = \sum_{i=1}^{n} B_n \cdots B_{i+1} [A, B_i] B_{i-1} \cdots B_1 \quad \cdots \quad (3)$$

Result (3) can be looked up in any Math Methods textbook.

Using (3) for $A = \hat{N}$ and $B_n \ldots B_1$ to be of ϕ & $\phi^\dagger$ ~~creation~~

we have

$$[\hat{N}, \phi^\dagger(x_1) \ldots \phi^\dagger(x_n)\, \phi(y_1) \ldots \phi(y_m)] =$$

$$= [N, \phi^\dagger(x_1)]\phi^\dagger(x_2)\cdots\phi^\dagger(x_n)\phi(y_1)\cdots\phi(y_m)$$

$$+ \phi^\dagger(x_1)[N, \phi^\dagger(x_2)]\cdots\phi^\dagger(x_n)\phi(y_1)\cdots\phi(y_m)$$

$$+ \cdots + \phi^\dagger(x_1)\cdots\phi^\dagger(x_n)\phi(y_1)\cdots\phi(y_{m-1})[N, \phi(y_m)]$$

Now using ① & ② we get,

$$= (n-m)\phi^\dagger(x_1)\phi^\dagger(x_2)\cdots\phi^\dagger(x_n)\phi(y_1)\cdots\phi(y_m)$$

This is a very powerful general result.

$$H = \hat{T} + \hat{V}$$

$$= \int d^3x\, \phi^\dagger(x)T(x)\phi(x) + \frac{1}{2}\int d^3x \int d^3y\, \phi^\dagger(x)\phi^\dagger(y)V(x,y)\phi(x)\phi(y)$$

$$[N,H] = [N,T] + [N,V]$$

→ not quite

$$\int d^3x\,(1-1)\phi^\dagger(x)\,T(x)\,\phi(x) + [N,V]$$

So value for $\hat{T}$: $n=1$; $m=1$

and for $\hat{V}$: $n=2$; $m=2$

similar here

$$= 0 + (2-2)\phi^\dagger(x)\phi^\dagger(y)\phi(x)\phi(y)$$

$$= 0$$

✓

Clearly we have now $[N, H] = 0$ ✓

3. $\langle \vec{k}_1' \bar{k}_2' | \hat{V} | \bar{k}_1 \bar{k}_2 \rangle$

$= \langle 0 | b(k_2') b(k_1') \hat{V} b^+(k_1) b^+(k_2) | 0 \rangle$

where $|\bar{k}_a \bar{k}_b\rangle = b^+(\bar{k}_a) b^+(\bar{k}_b)|0\rangle$ is the 2-particle state.

$V(\bar{x}, \bar{y}) = \dfrac{e^2}{|\bar{x} - \bar{y}|}$

now $\hat{v}$ is given by Problem 2, which is

$|\bar{k}_1 \bar{k}_2\rangle = b^+(\bar{k}_1) b^+(\bar{k}_2)|0\rangle$

$$\hat{V} = \frac{1}{2} \int d^3x \int d^3y \, \phi^+(x) \phi^+(y) \frac{e^2}{|\bar{x} - \bar{y}|} \phi(x) \phi(y)$$

$\langle \bar{k}_1' \bar{k}_2' | = \langle 0 | b(k_2') b(k_1')$

So we have

$$Z = \langle \bar{k}_1' \bar{k}_2' | \hat{V} | \bar{k}_1 \bar{k}_2 \rangle$$

$$= \langle 0 | b(k_2') b(k_1') \frac{1}{2} \int d^3x \int d^3y \, \phi^+(x) \phi^+(y) \frac{e^2}{|\bar{x} - \bar{y}|} \phi(x) \phi(y) b^+(k_1) b^+(k_2) | 0 \rangle$$

Now using the fact that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} b(\bar{k}) e^{i\bar{k} \cdot \bar{x}}$$

$$\phi^+(x) = \int \frac{d^3k}{(2\pi)^3} b^+(\bar{k}) e^{-i\bar{k} \cdot \bar{x}}$$

$$Z = \langle 0 | b(k_2')b(k_1') \frac{1}{2}\int d^3x \int d^3y \int d^3k \frac{1}{(2\pi)^3} b^\dagger(k) e^{-ik|\bar{x}-\bar{y}|} \frac{e^2}{|\bar{x}-\bar{y}|} \cdots |0\rangle$$

$|\bar{x}-\bar{y}| = |\bar{r}|$, and using $\int d^3x \, e^{i(\bar{k}'-\bar{k})x} = (2\pi)^3 \delta^3(k-\bar{k}')$

we get ; [also use $\vec{q} = (\bar{k}_1+\bar{k}_2) - (\bar{k}_3 - \bar{k}_4)$

→ exchange momentum
= 0 by mom cons?

$$Z = \int d^3\bar{r} \, \frac{e^2}{|\bar{r}|} e^{i\bar{q}\cdot\bar{r}}$$

Going to spherical coordinates

$$Z = 2\pi \int_0^\infty r^2 dr \frac{e^2}{|\bar{r}|} e^{iqr\cos\theta} \int_{-1}^{1} d(\cos\theta)$$

$$= 2\pi e^2 \int_0^\infty \frac{r^2}{|\bar{r}|} dr \int_{-1}^{1} d(\cos\theta) \, e^{iqr\cos\theta}$$

Regulate the integral by multiplying $e^{-\xi r}$ ✓

and take $\xi \to 0$

$$= 2\pi e^2 \int_0^\infty \frac{r^2 e^{-\xi r}}{r} \int_{-1}^{1} d(\cos\theta) \, e^{iqr\cos\theta}$$

$$= \frac{4\pi e^2}{q^2}$$ ✓

✓ not quite, but right idea
see solutions

Another way to do the integral is

$$Z = \int d^3\bar{r}\ \frac{e^2}{|\bar{r}|}\ e^{i\bar{q}\cdot\bar{r}}$$

$$= 2\pi e^2 \int_0^\infty dr \int_{-1}^{1} d(\cos\theta)\, e^{iqr\cos\theta}\, r$$

$$= \frac{4\pi e^2}{q} \int_0^\infty \sin(rq)$$

$$= \frac{4\pi e^2}{q} \left| -\cos(rq) \right|_0^\infty$$

$$= \frac{4\pi e^2}{q^2} \quad ✓.$$

$$\left\{ \begin{array}{l} \lim_{x\to 0} \frac{\sin x}{x} = \phi \\ \lim_{x\to \pi/2} \frac{\cos x}{x} = 1 \end{array} \right.$$

↳ Slightly hand-waving.

4. $[\hat{N}, \phi(x)]$

where $N = \int d^3x' \, \phi^+(x') \, \phi(x')$

$$[\hat{N}, \phi(x)] = \left[\int d^3x' \, \phi^+(x') \phi(x'), \; \phi(x) \right] \quad \cdots \quad (1)$$

using $[AB, C) = A[B,C] + [A,C]B$ in (1)

$$= \left(\int d^3x' \, \phi^+(x') [\phi(x'), \phi(x)] + \int d^3x' \, [\phi^+(x'), \phi(x)] \phi(x') \right.$$

$$[\phi(x'), \phi(x)] \to 0, \qquad [\phi^+(x'), \phi(x)] \to -\delta(x'-x)$$

$$= \int d^3x' \, -\delta(x'-x) \, \phi(x')$$

$$= -\phi(x)$$

ii) $[H, \phi(x)]$ where $H = \hat{T} + \hat{V}$

$$= \int d^3x \, \phi^+(x) T(x) \phi(x)$$

$$+ \frac{1}{2} \int d^3x \int d^3y \, \phi^+(x) \phi^+(y) V(x,y) \phi(x) \phi(y)$$

$$[H, \phi(x)] = [T, \phi(x)] + [V, \phi(x)]$$

Consider first $[T, \phi(x)]$

$$[T, \phi(x)] = \int d^3x' \left[\phi^\dagger(x') T \phi(x'), \phi(x)\right]$$

$$= \int d^3x' \left\{ \phi^\dagger(x') \left[T\phi(x'), \phi(x)\right] \right. \rightarrow \text{ZERO}$$

$$+ \int d^3x' \left[\phi^\dagger(x'), \phi(x)\right] T\phi(x')$$

$$= \int d^3x' \, -\delta(x'-x) \, T \, \phi(x')$$

$$= -T\phi(x) \Big|_{\text{evaluated at } x}$$

Now consider $[V, \phi(x)]$

$$[V, \phi(x)] = \frac{1}{2} \int d^3x \, d^3y \left[\phi^\dagger(x)\phi^\dagger(y) V \phi(x) \phi(y), \phi(x)\right]$$

using the commutator identity

$$[ABCD, E] = ABC[D,E] + AB[C,E]D + A[B,E]CD + [A,E]BCD$$

we get,

$$[V, \phi(x)] = \frac{1}{2} \int d^3x' d^3y \, [\phi(y), \phi(x)] \, \phi^\dagger(x') \phi^\dagger(y) V \phi(x')$$

$$+ \frac{1}{2} \int d^3x' d^3y \, \phi^\dagger(x) \phi^\dagger(y) \, [\phi(x'), \phi(y)] V \, \phi(x)$$

$$+ \frac{1}{2} \int d^3x' d^3y \, \phi^\dagger(x') [\phi^\dagger(y), \phi(x)] V \phi(x') \phi(y)$$

$$+ \frac{1}{2} \int d^3x' d^3y \, [\phi^\dagger(x'), \phi(x)] \phi^\dagger(y) V \phi(x') \phi(y) \quad \ldots (2)$$

Now, since $[\phi(y), \phi(x)] = 0$ and $[\phi(x'), \phi(y)] = 0$

The first two term goes to zero.

Also $[\phi^{+}(x'), \phi(x)] = -\delta^3(x'-x)$

$$[\phi^{+}(y), \phi(x)] = -\delta^3(x-y) \quad \cdots (3)$$

Use (3) in (2), and integrate δ^3 over all volume to get

$$[V, \phi(x)] = -V\,\phi(x)$$

So, $[H, \phi(x)] = -(T+V)\phi(x)$

$$= -(T_x + V_x)\phi(x)$$

Hence $\phi(x)$ lowers the energy at 'x'.

$$[H, \phi^{+}(x)] = T_x + V_x\,\phi(x)$$

↳ raises the energy at x

$$\Psi_N(x_1 \ldots x_N; t) = \frac{1}{\sqrt{N!}} \langle 0 | \phi(x_1) \ldots \phi(x_N) | \Psi_N(t) \rangle$$

Satisfies

a) $$\int \left(\prod_{i=1}^{N} d^3 x_i \right) \left| \Psi_N(x_1 \ldots x_N; t) \right|^2 = 1$$

Proof :

$$\frac{1}{N!} \int \left(\prod_{i=1}^{N} d^3 x_i \right) \langle \Psi_N(t) | \phi^\dagger(x_N) \ldots \phi^\dagger(x_1) | 0 \rangle \langle 0 | \phi(x_1) \ldots \phi(x_N) | \Psi_N(t) \rangle \quad \ldots (1)$$

Now the knowledge that N-particle state forms a complete set will help us

$$\phi^\dagger(x_N) \ldots \phi^\dagger(x_1) | 0 \rangle = | x_N, \ldots x_1 \rangle \quad \ldots (2)$$

And $|x_1 \ldots x_N\rangle$ forms a complete set

$$\int d^3x_1 \ldots d^3x_N \, | x_1 \ldots x_N \rangle \langle x_1, \ldots x_N | = N! \quad \ldots (3)$$

↓ since we have $N!$ permutations

Ex: In case of $n = 3$, we have 6 possiblities

$|x_1, x_2, x_3\rangle$ $|x_1, x_3, x_2\rangle$

$|x_2, x_1, x_3\rangle$ $|x_2, x_3, x_1\rangle$ ⟶ hence $N!$

$|x_3, x_1, x_2\rangle$ $|x_3, x_2, x_1\rangle$

a. Using ③ in ① gives,

$$\frac{1}{N!}\int \prod_{i=1}^{N} d^3x_i \;\tilde{\xi}\; \langle \Psi_N(t)|\Psi_N(t)\rangle$$ where $\tilde{\Sigma} = |x_1 \ldots x_N\rangle\langle x_1 \ldots x_N|$

$$\frac{1}{N!}\; N!\; \langle \Psi_N(t)|\Psi_N(t)\rangle = 1$$

✓

b) $$i\frac{\partial}{\partial t}\Psi_N(x_1\ldots x_N;t) = \left[\sum_{k=1}^{N} \frac{-1}{2m}\nabla_k^2 + \frac{1}{2}\sum_{k\neq l} V(x_k, x_l)\right]\Psi_N(x_1\ldots x_N;t)$$

$$\hat{H}\,\Psi_N(x_1\ldots x_N;t) = E\,\Psi_N(x_1\ldots x_N;t) \quad \cdots \quad ①$$
$$= i\hbar\frac{\partial}{\partial t}\Psi_N(x_1\ldots x_N;t)$$

Let's act the $\hat{H}$ operator on Ψ_N given in the problem, then ① becomes

$$E\Psi_N(x_1\ldots x_N;t) = \frac{1}{N!}\langle 0|\phi(x_1)\ldots\phi(x_N)\hat{H}|\Psi_N(t)\rangle \quad \cdots ②$$

Now we use the commutator

$$[H, \phi(x)] = -(T+V)\phi(x)$$

$$H\phi(x) - \phi(x)H = -(T+V)\phi(x)$$

$$\phi(x)H = H\phi(x) + (T+V)\phi(x) \quad \cdots ③$$

using ③ in ② to pull the $\hat{H}$ all the way left and noting that $\langle 0 | \hat{H} = \langle 0 |$ we get, using ①, ② & ③

$$i\hbar \frac{\partial}{\partial t} \psi_N(x_1 \ldots x_N; t) = \left| \hat{T} + \hat{V} \right|_{\substack{\text{evaluated} \\ \text{at all} \\ x_1 \ldots x_N}} \psi_N(x_1 \ldots x_N; t)$$

$$= \left[-\sum_{k=1}^{N} \frac{\nabla_k^2}{2m} + \frac{1}{2} \sum_{k \neq l} V(x_k, x_l) \right] \psi_N(x_1 \ldots x_N; t)$$

✓

✓

6. a) $$\lim_{\epsilon \to 0^+} \frac{1}{x+i\epsilon} = P\frac{1}{x} - i\pi\delta(x)$$

To check this, consider

$$f(x) = \lim_{\epsilon \to 0+} \frac{g(x)}{x+i\epsilon} = \lim_{\epsilon \to 0+} \frac{x\,g(x)}{x^2+\epsilon^2} - \frac{i\epsilon\, g(x)}{x^2+\epsilon^2}$$

For $x \neq 0$, $f(x) = \frac{g(x)}{x}$
For $x = 0$, $f(x) \longrightarrow -i\infty$
} Conditions

If we integrate $f(x)$ for $a < 0 < b$, we get

$$\int_a^b dx\, f(x) = PV\int_a^b dx\, \frac{g(x)}{x} - i \lim_{\epsilon \to 0+} \left[\tan^{-1}\left(\frac{b}{\epsilon}\right) - \tan^{-1}\left(\frac{a}{\epsilon}\right)\right] g(x)$$

[Since Integral of that term is $\tan^{-1}$]

$$= PV\int_a^b dx\, \frac{g(x)}{x} - i\pi$$

i.e

$$\int_a^b dx\, \frac{g(x)}{x+i\epsilon} = PV\int_a^b dx\, \frac{g(x)}{x} - i\pi\, g(0)$$

Also the second term $\longrightarrow 0$ as $x \to 0$

We find a function that gives $-i\pi$ when integrated and ∞ when taken at 0.

We are reminded of δ-function.

$$\underset{\epsilon\to 0+}{Lt}\int dx \frac{g(x)}{x+i\epsilon} = PV\int_a^b dx \frac{g(x)}{x} - i\pi\int_a^b \delta(x) g(x)\, dx$$

we read off the integrand as

$$\underset{\epsilon\to 0+}{Lt} \frac{g(x)}{x+i\epsilon} = PV \frac{g(x)}{x} - i\pi\, g(x)\,\delta(x)$$

Since $g(x)$ was our smooth f^n, we have

$$\underset{\epsilon\to 0+}{Lt} \frac{1}{x+i\epsilon} = PV \frac{1}{x} - i\pi\,\delta(x).$$

✓ ✓

a) F.T of $\theta(x)$

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

We can write this as

$$\theta_\epsilon(x) = \begin{cases} e^{-\epsilon x} & x \geq 0 \\ 0 & x < 0 \end{cases} \Bigg| \text{as } \epsilon \to 0$$

$$\tilde{f}(k) = \int dx \; e^{-(ik+\epsilon)x} dx$$

$$= \int_0^\infty dx \; e^{-(ik+\epsilon)x} dx = \frac{1}{ik+\epsilon}$$

Now multiply by $\epsilon - ik$, we get

$$\Rightarrow \frac{1}{ik+\epsilon} \; \frac{\epsilon - ik}{\epsilon - ik} = \frac{\epsilon - ik}{\epsilon^2 + k^2}$$

$$= \frac{\epsilon}{\epsilon^2+k^2} - \frac{ik}{\epsilon^2+k^2}$$

Now, take the limit of $\epsilon \to 0$

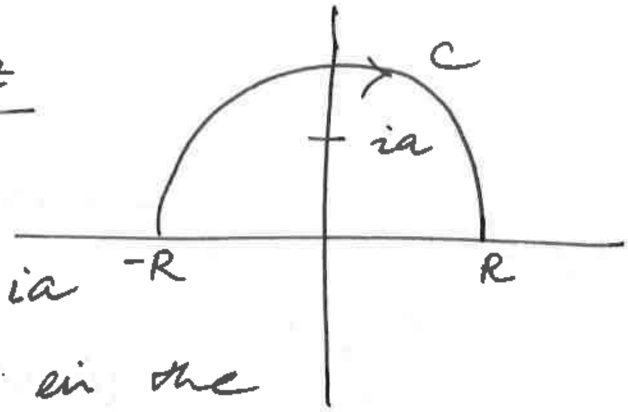
$$= \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon^2+k^2} - \frac{i}{k}$$ ✓

$$= \pi \, \delta(k) + P\frac{1}{ik}$$ [using the work done in PART ①]

✓ Ans.

b) F.T of $\dfrac{1}{x^2+a^2}$

Consider $f(z) = \oint \dfrac{e^{ikz}\,dz}{z^2+a^2}$



The poles are at $z = \pm ia$
but since our contour is in the upper half plane. only $z = +ia$ lies in it

Residue at $z = ia \Rightarrow \lim_{z \to ia} \dfrac{(z-ia)\, e^{ikz}}{(z-ia)(z+ia)}$

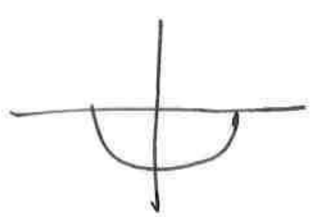
$$= \frac{e^{-ka}}{2ia}$$

Residue theorem and take the limit $R \to \infty$ (contribution of the upper curve $\to 0$)

$$\int_{-\infty}^{\infty} dx \frac{e^{ikx}}{x^2+a^2} = i 2\pi \frac{e^{-ak}}{2ia}$$

$$= \frac{\pi}{a} e^{-ak}$$

when $k > 0$



When $k < 0$, we consider lower contour and integral takes the value

$$(\pi/a) e^{ak}$$

Putting together,

$$\int_{-\infty}^{\infty} dx \, \frac{e^{ikx}}{x^2+a^2} = \frac{\pi}{a} e^{-a|k|}$$ ✓ ✓

———————— x ————————

(5) Compton wavelength of proton $\approx$ 1.3 fm

$\approx 1.3 \times 10^{-15}$ m

Two ways to this problem

(1) Consider $\Delta E \cdot \Delta t \approx \hbar$

$$\Delta E \cdot \frac{\Delta x}{c} \approx \hbar$$

$$\Delta x \approx \frac{6.6 \times 10^{-34} \times 3 \times 10^{8}}{10^{12} \times 1.6 \times 10^{-19}}$$ ✓

$\approx 10^{-19}$ m

(2) We know that when we work in godforsaken units $\hbar = c = 1$

Also $\hbar c = 197.3$ MeV fm

Since 10^{12} eV $= 10^{6}$ MeV ..

$$\Delta x \Big|_{\text{corresponding to 1 TeV}} = \frac{197.3}{10^{6}} \text{ fm}$$

$\approx 10^{-4}$ fm

$\approx 10^{-19}$ m

} Matches ...

Now

$$\frac{\Delta x \big|_{\text{at LHC}}}{\lambda_{\text{compton}/\text{proton}}} = \frac{10^{-19}\ m}{1.3 \times 10^{-15}\ m} = 10^{-4}$$

$$\left\{ \frac{\lambda_{\text{comp}}}{\Delta x_{LHC}} = 10^4 \right.$$

LHC at 1 TeV probes to distances much smaller that compton wavelength!

7. a. $I(\alpha) = \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\alpha} f(x)\right\}$

$f(x)$ has a unique global minimum at $x = 0$

$$-\frac{1}{\alpha} f(x) = -\frac{1}{\alpha} f(0) - \frac{1}{\alpha} \frac{f'(0)\,x}{1!} - \frac{1}{\alpha} \frac{f''(0)\,x^2}{2!} - \frac{1}{\alpha} \frac{f'''(0)\,x^3}{3!}$$

$\longrightarrow$ ZERO

Now $f'(0) = 0 \longrightarrow$

$$= -\frac{1}{\alpha} f(0) - \frac{1}{\alpha} \frac{f_0^{(2)}}{2!} x^2 - \sum_{k=3}^{\infty} \frac{1}{\alpha} \frac{f_0^{(k)}}{k!} x^k \quad \ldots \text{(1)}$$

Now let's make a substitution $x \longrightarrow \sqrt{\alpha}\, y$

(1) takes the form

$$= -\frac{1}{\alpha} f(0) - \frac{1}{\alpha} \frac{f_0^{(2)} \alpha y^2}{2!} - \sum_{k=3}^{\infty} \frac{\alpha^{k/2-1} f_0^{(k)}}{k!} y^k$$

Let's take $\xi(y) = \sum_{k=3}^{\infty} \frac{\alpha^{k/2-1} f_0^{(k)} y^k}{k!}$

$$= -\frac{1}{\alpha} f(0) - \frac{1}{2!} f_0^{(2)} y^2 - \xi(y)$$

Putting this in the integral

$dx = \sqrt{\alpha}\, dy$

$$I(\alpha) = \int_{-\infty}^{\infty} \exp\left(\frac{-f_0}{\alpha} - \frac{1}{2!} f_0^{(2)} y^2 - \varepsilon(y) \right) dy \sqrt{\alpha}$$

$$= e^{\frac{-f_0}{\alpha}} \sqrt{\alpha} \int_{-\infty}^{\infty} \exp\left(-\frac{f_0^{(2)} y^2}{2!} - \varepsilon(y) \right) dy$$

$$= e^{\frac{-f_0}{\alpha}} \sqrt{\alpha} \int_{-\infty}^{\infty} \exp\left(\frac{-f_0^{(2)} y^2}{2} \right) \exp\left(-\varepsilon(y) \right) dy$$

Since, we want to expand in terms till first order of α, we can write

$$\exp(-\varepsilon(y)) \cong 1 - \varepsilon(y) + \frac{[\varepsilon(y)]^2}{2!}$$

$$= e^{\frac{-f_0}{\alpha}} \sqrt{\alpha} \int_{-\infty}^{\infty} \exp\left(-\frac{f_0^{(2)} y^2}{2} \right) \left[1 - \varepsilon(y) \right] dy$$

Now pull the first term by integrating Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

$$= e^{\frac{-f_0}{\alpha}} \sqrt{\alpha} \sqrt{\frac{2\pi}{f_0^{(2)}}} \left[1 + \int_{-\infty}^{\infty} - \varepsilon(y) \right]$$

$$= \sqrt{\alpha}\, e^{-\frac{f_0}{\alpha}} \int_{-\infty}^{+\infty} e^{-\frac{f_2 y^2}{2!}} \left[1 - \frac{f_4 y^4 \alpha}{4!} + \frac{f_6 y^6 \alpha}{6!} \right.$$

$$= e^{-\frac{f_0}{\alpha}} \sqrt{\frac{2\pi\alpha}{f^{(2)}}} \left[1 + \int_{-\infty}^{\infty} e^{-\frac{f^{(2)} y^2}{2!}} \frac{f_6 y^6 \alpha}{6!} - \int_{-\infty}^{\infty} e^{-\frac{f^{(2)} y^2}{2!}} \frac{f_4 y^4 \alpha}{4!} \right]$$

(*)

$$= e^{-\frac{f_0}{\alpha}} \sqrt{\frac{2\pi\alpha}{f^{(2)}}} \left\{ 1 + \frac{f_6 \alpha}{6!} \frac{2\Gamma(7/2)}{\left[\frac{f^{(2)}}{2}\right]^{7/2}} - \frac{f_4 \alpha}{4!} \frac{2\Gamma(5/2)}{\left[\frac{f^{(2)}}{2}\right]^{5/2}} \right.$$

TRIED AGAIN AT END!!

$$\left\{ 1 + \frac{(f_0^{(3)})^2}{120 \times 6} \frac{2 \times 10/3\, \alpha}{f^2} - \frac{f_4 \alpha}{4!} \frac{2 \times 4/3}{[f^{(2)}]^2 \frac{1}{4}} \right.$$

$$\left\{ 1 + \right.$$

. . ?

$$= e^{-\frac{f_0}{\alpha}} \sqrt{\frac{2\pi\alpha}{f^{(2)}}} \left\{ 1 + \left[\frac{(f_0^{(3)})^2}{120 \times 6} \frac{2 \times 10/3}{[f^{(2)}]^2} - \frac{f_4}{4!} \frac{8/3}{[f^{(2)}]^2 \frac{1}{4}} \right] \alpha + O(\alpha^2) \right\}$$

(*) we used $\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \dfrac{2\Gamma\left(\frac{n+1}{2}\right)}{(a)^{\frac{n+1}{2}}}$

Integral can be brought to this form by assuming $x = \sqrt{\alpha}\, y$ and expanding to order of α.

b) Now $\alpha \to i\alpha$, the derived expression will remain the same.

$$I(i\alpha) = e^{-f_0/i\alpha}\sqrt{\frac{2\pi i\alpha}{f_0^{(2)}}}\left\{1 + \left(\frac{5}{24}\frac{(f_0^{(3)})^2}{(f_0^{(2)})^3} - \frac{3}{24}\frac{f_0^{(4)}}{(f_0^{(2)})^2}\right) + O(\alpha^2)\right.$$

Since, the result we derived was under the assumption of $\alpha \to 0$, $i \to i\alpha$ is justified. The phases will oscillate very rapidly ($\alpha \to 0$) and will cancel the contributions and the expansion will remain valid there also.

This is what we do in Path Integrals where the role of 'α' is played by $\hbar$

$e^{\frac{iS}{\hbar}}$ integrals valid under $\xi \to iS$ ✓.

See soln for justification of analytic continuation in α

c. $U(t) = \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle$

use $H = E = \sqrt{\vec{p}^{\,2} + m^2}$

$$U(t) = \langle \vec{x} | e^{-it\sqrt{\vec{p}^2+m^2}} | \vec{x}_0 \rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3p \, e^{-it\sqrt{p^2+m^2}} \cdot e^{ip\cdot(\bar{x}-\bar{x}_0)}$$

Now we can write $\vec{p}\cdot(\vec{x}-\vec{x}_0) = p|x-x_0|\cos\theta$

Also, we go to sp. polar coordinates

$$U(t) = \frac{1}{(2\pi)^3} \int_0^\infty p^2\, dp \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, \exp\left(ip|x-x_0|\cos\theta\right) \exp\left(-it\sqrt{p^2+m^2}\right)$$

Doing the 'θ' & ϕ integrals first we get

$$= \frac{1}{4\pi^2} \frac{1}{i|\bar{x}-\bar{x}_0|} \int_0^\infty p\, dp \left[e^{ip|x-x_0|} - e^{-ip|x-x_0|} \right] e^{-it\sqrt{p^2+m^2}}$$

$$= \frac{1}{4\pi^2} \frac{1}{i|\bar{x}-\bar{x}_0|} \int_0^\infty p\, dp\, \sin(p|x-x_0|) . 2i\, e^{-it\sqrt{p^2+m^2}}$$

$$= \frac{1}{2\pi^2|\bar{x}-\bar{x}_0|} \int_0^\infty dp\, p \sin(p|x-x_0|) e^{-it\sqrt{p^2+m^2}} \quad \ldots \quad (2)$$

$\vec{x}^2 \gg t^2$

Let $|x - x_0| = |x|$

The phase function is $px - t\sqrt{p^2+m^2}$

↓ from sin

↓ from e^{-iHt}

Now, the point of interest for the phase function (stationary point) is $p = \frac{imt}{\sqrt{x^2-t^2}}$

Plug this value in ② to get

$$U(t) \sim e^{-m\sqrt{x^2-t^2}}$$

↓

This is clearly non-zero for any value of x and t. This violates Causality, since the speed of propagation can't exceed that of c.

8) a $J^{\mu\nu} = L^{\mu\nu}$ with $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$

defined. $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$

$$[J^{\mu\nu}, J^{\rho\sigma}] = [L^{\mu\nu}, L^{\rho\sigma}]$$

$$= [i(x^\mu\partial^\nu - x^\nu\partial^\mu), i(x^\rho\partial^\sigma - x^\sigma\partial^\rho)]$$

$$= -(x^\mu g^{\nu\rho}\partial^\sigma - x^\rho g^{\mu\sigma}\partial^\nu - x^\nu g^{\mu\rho}\partial^\sigma + x^\rho g^{\nu\sigma}\partial^\mu + x^\nu g^{\mu\sigma}\partial^\rho - x^\sigma g^{\nu\rho}\partial^\mu - x^\mu g^{\nu\sigma}\partial^\rho + x^\sigma g^{\mu\rho}\partial^\nu) \quad ⊛$$

$$= i(g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\nu\rho} - g^{\mu\rho}L^{\nu\sigma} - g^{\nu\sigma}L^{\mu\rho})$$

---

where in the ⊛ step, we have used the result that $\partial^\mu x^\nu = g^{\mu\nu}$

8.b) $$[J^{\mu\nu}, J^{\rho\sigma}]^\kappa{}_\lambda = (J^{\mu\nu})^\kappa{}_\gamma (J^{\rho\sigma})^\gamma{}_\lambda - (J^{\rho\sigma})^\kappa{}_\gamma (J^{\mu\nu})^\gamma{}_\lambda$$

$$= i(\delta_\mu{}^\kappa g_{\nu\gamma} - \delta_\nu{}^\kappa g_{\mu\gamma})(\delta_\rho{}^\gamma g_{\sigma\lambda} - \delta_\sigma{}^\gamma g_{\rho\lambda}) - (\delta_\rho{}^\kappa g_{\sigma\gamma} - \delta_\sigma{}^\kappa g_{\rho\gamma})(\delta_\mu{}^\gamma g_{\nu\lambda} - \delta_\nu{}^\gamma g_{\mu\lambda})$$

$$= i g_{\nu\rho}(\delta^\kappa_\mu g_{\sigma\lambda} - \delta^\kappa_\sigma g_{\mu\lambda}) + \text{other permutations} \; ---$$

$$= i g_{\nu\rho}(J_{\mu\sigma})^\kappa{}_\lambda + \; ---$$

} Later on next page

$$= ig^{\nu\rho}(J^{\mu\sigma})^{\kappa}{}_{\lambda} + ig^{\mu\sigma}(J^{\nu\rho})^{\kappa}{}_{\lambda} - ig^{\mu\rho}(J^{\nu\sigma})^{\kappa}{}_{\lambda} - ig^{\nu\sigma}(J^{\mu\rho})^{\kappa}{}_{\lambda}$$

✓

8.c. $J^{\mu\nu} = S^{\mu\nu}$ with $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, where

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}$$

$$\left.\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu}\mathbb{I} \\ [\gamma^\mu, \gamma^\nu] &= \frac{4S^{\mu\nu}}{i} \end{aligned}\right\}$$

using $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I}$, we get

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]$$

$$= g^{\mu\nu}\mathbb{I}_{4\times4} - 2iS^{\mu\nu}$$

Then, since $g^{\mu\nu}\mathbb{I}_{4\times4}$ commutes with any matrix X, we have

$$[X, S^{\mu\nu}] = \frac{i}{2}[X, \gamma^\mu\gamma^\nu] \quad \dots \quad ①$$

Now, we consider $[\gamma^\rho\gamma^\sigma, S^{\mu\nu}] = \gamma^\rho[\gamma^\sigma, S^{\mu\nu}] + [\gamma^\rho, S^{\mu\nu}]\gamma^\sigma$

$$= \gamma^\rho[\gamma^\sigma, S^{\mu\nu}] + [\gamma^\rho, S^{\mu\nu}]\gamma^\sigma \quad \dots \quad ②$$

$$[\gamma^\sigma, S^{\mu\nu}] \overset{\text{using ①}}{=} \frac{i}{2}[\gamma^\sigma, \gamma^\mu\gamma^\nu]$$

$$= \frac{i}{2}\{\gamma^\sigma, \gamma^\mu\}\gamma^\nu - \frac{i}{2}\gamma^\mu\{\gamma^\sigma, \gamma^\nu\}$$

$$= \frac{i}{2}2g^{\sigma\mu}\gamma^\nu - \frac{i}{2}2g^{\sigma\nu}\gamma^\mu$$

$$= ig^{\sigma\mu}\gamma^\nu - ig^{\sigma\nu}\gamma^\mu \quad \dots \quad ③$$

using

$$[A, BC] = [A, B]C + B[A, C]$$

$$= \{A, B\}C - B\{A, C\}$$

using ③ in ② we get,

$$= \gamma^\rho(ig^{\sigma\mu}\gamma^\nu - ig^{\sigma\nu}\gamma^\mu) + (ig^{\rho\mu}\gamma^\nu - ig^{\rho\nu}\gamma^\mu)\gamma^\sigma$$

$$= ig^{\sigma\mu}\gamma^\rho\gamma^\nu - ig^{\rho\nu}\gamma^\mu\gamma^\sigma - ig^{\sigma\nu}\gamma^\mu\gamma^\rho + ig^{\rho\mu}\gamma^\nu\gamma^\sigma$$

$$= ig^{\sigma\mu}(g^{\rho\nu} - 2iS^{\rho\nu}) - ig^{\rho\nu}(g^{\mu\sigma} + 2iS^{\mu\sigma}) - ig^{\sigma\nu}(g^{\mu\rho} - 2iS^{\mu\rho}) + ig^{\rho\mu}(g^{\nu\sigma} + 2iS^{\nu\sigma})$$

$$= 2g^{\sigma\mu}S^{\rho\nu} - 2g^{\rho\nu}S^{\mu\sigma} - 2g^{\sigma\nu}S^{\mu\rho} + 2g^{\rho\mu}S^{\nu\sigma}$$

Hence, we finally use

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{i}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma]$$

$$= i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

---

b) $$U = \exp(-i/2\,\omega_{\mu\nu} J^{\mu\nu})$$

Given $\omega_{12} = -\omega_{21} = \theta$

$$\mathcal{J}^{12} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, exponential of matrix is :

$$e^A = 1 + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$$

$$-\frac{i}{2}\omega_{12} J^{12} = i\,\frac{-i}{2}\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A$$

$$U = \exp(A) = 1 + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\theta^2 & 0 & 0 \\ 0 & 0 & -\theta^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\cdot\frac{1}{2}$$

$$+ \frac{1}{12}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^3 & 0 \\ 0 & -\theta^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\cdot\frac{1}{4}$$

required explicit answer.

✓

Some factor.

Some factor.

$$= (\mathbb{1})_{4\times4} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\theta & 0 \\ 0 & \sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

✗

~~Can be written in form~~ of sin & cosine. !!

$$\begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}$$

- $\omega_{01} = -\omega_{10} = \beta$

$$J^{01} = i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \exp\left(\frac{-i}{2} \omega_{01} J^{01}\right)$$

$$= \exp\left(\frac{-i}{2} \beta J^{01}\right)$$

$$= \exp\left(\frac{-i}{2} \beta J^{01}\right) = \exp\left(\frac{1}{2}\begin{pmatrix} 0 & \beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)$$

$$U = \exp(A)$$

where $A = \begin{pmatrix} 0 & \beta/2 & 0 & 0 \\ \beta/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$U = 1_{4\times4} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \beta/2 & 0 & 0 \\ \beta/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \beta^2/4 & 0 & 0 & 0 \\ 0 & \beta^2/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & \beta/\text{[illegible]} & 0 & 0 \\ \beta/\text{[illegible]} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \frac{1}{8} \begin{pmatrix} \beta^2/\text{[illegible]} & 0 & 0 & 0 \\ 0 & \beta^2/\text{[illegible]} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{48} \begin{pmatrix} 0 & \beta^3/\text{[illegible]} & 0 & 0 \\ \beta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\longrightarrow$

cosh & sinh

Starting Problem 7 again from mid-way

$$\xi(y) = \sum_{k=3}^{\infty} \frac{f_0^{(k)} y^k \alpha^{k/2-1}}{k!}$$

we had

$$I(\alpha) = e^{-f_0/\alpha} \sqrt{\frac{2\pi\alpha}{f^{(2)}}} \left\{ 1 - \int_{-\infty}^{\infty} e^{-\frac{f^{(2)} y^2}{2!}} \xi(y)\, dy + \text{Terms from } [\xi(y)]^2 \right\}$$

$$= e^{-f_0/\alpha} \sqrt{\frac{2\pi\alpha}{f^{(2)}}} \left\{ 1 - \left(\frac{3\sqrt{\pi}}{4\left(\frac{f^{(2)}}{2}\right)^{5/2}} \frac{f^{(4)}}{4!} \sqrt{\frac{f_2}{2\pi}} \right) \alpha + \text{Terms} \right\}$$

$$= e^{-f_0/\alpha} \sqrt{\frac{2\pi\alpha}{f^{(2)}}} \left\{ 1 - \left(\frac{3}{4!} \frac{f^{(4)}}{f^{(2)}} \right) \alpha + \text{Terms} \right\}$$

Now Terms corresponding to $\frac{[\xi(y)]^2}{2!}$ terms.

$$\text{Terms} = \int_{-\infty}^{\infty} \left(e^{-\frac{f^{(2)} y^2}{2!}} \frac{(f_0^{(3)})^2 y^6}{(3!)^2} \right) \alpha\, dy$$

Evaluating $\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{(a)^{n+1/2}}$

$$I(\alpha) = e^{-f_0/\alpha}\sqrt{\frac{2\pi\alpha}{f_2}}\left\{1 + \left[\frac{\cancel{8}\times\cancel{15}}{2\times\cancel{8}\times(6)^2}\frac{(f^{(3)})^2}{(f^{(2)})^3} - \frac{3}{24}\frac{f^{(4)}}{(f^{(2)})^2}\right]\alpha + O(\alpha^2)\right]$$

$$15/72$$

$$= e^{-f_0/\alpha}\sqrt{\frac{2\pi\alpha}{f^{(2)}}}\left\{1 + \left[\frac{5}{24}\frac{(f^{(3)})^2}{(f^{(2)})^3} - \frac{3}{24}\frac{f^{(4)}}{(f^{(2)})^2}\right]\alpha + O(\alpha^2)\right]$$

✓

Took almost $4\frac{1}{2}$ hours to fix all number.

Feels good.

Some exercises are designed (in part) to build calculational fortitude

Q.F.T Assignment - 2

RAGHAV GOVIND JHA

11 February 2014

60/60

Problem # 1 Casimir Effect

a) $$E_0 = \langle 0|H|0\rangle = \frac{V}{2}\int \frac{d^3k}{(2\pi)^3}\sum_{K=1,2}\hbar\omega_K$$

$$= \hbar c\,\frac{L^2 a}{(2\pi)^3}\int d^2k_\perp \int dk_z\,|\bar{k}|$$

The boundary conditions imply that z- component of $\vec{k}$ i.e k_z is quantized and take values $k_z = \frac{n\pi}{a}$.

$$\mathcal{E}_0 = \frac{\Delta E_0}{L^2} = \frac{1}{L^2}\left(E_{free} - E_{boundary}\right)$$

We have $k_z = \frac{n\pi}{a}$

$$dk_z = \frac{\pi}{a}\,dn$$

$$E_k = \frac{\hbar c\,L^2 a}{(2\pi)^3}\int d^2k_\perp \int \frac{\pi}{a}\,dn\,\sqrt{(k_\perp)^2 + \left(\frac{n\pi}{a}\right)^2}$$

Enforcing boundary mean $\int_{-\infty}^{\infty} \longrightarrow \sum_{n=-\infty}^{n=\infty}$

$$E_{boundary} = \frac{\hbar c\,L^2 a}{(2\pi)^3}\int d^2k_\perp\left(\sum_{n=-\infty}^{n=\infty}\frac{\pi}{a}\sqrt{k_\perp^2 + \left(\frac{n\pi}{a}\right)^2}\right)$$

$$\mathcal{E}_0 = \frac{\Delta E_0}{L^2} = \frac{1}{L^2}\left\{ \frac{\hbar c L^2 a}{(2\pi)^3} \cdot \frac{\pi}{a} \int d^2k_\perp \left[\sum_{n=-\infty}^{n=\infty} - \int dn \right] \sqrt{k_\perp^2 + \left(\frac{n\pi}{a}\right)^2} \right\}$$

$$= \frac{\hbar c}{8\pi^2} \int d^2k_\perp \left(\sum_{n=-\infty}^{n=\infty} - \int_{-\infty}^{\infty} dn \right) \sqrt{k_\perp^2 + \left(\frac{n\pi}{a}\right)^2}$$

Now, we introduce a smooth cut-off $f(|\bar{k}|)$

$$f(|\bar{k}|) = \begin{cases} 1 & \text{if } |\bar{k}| < k \\ 0 & |\bar{k}| > k \end{cases}$$

Introduce $u = \frac{a^2}{\pi^2} k_\perp^2 + n^2$

$$k_\perp = \sqrt{\frac{\pi^2}{a^2}(u - n^2)}$$

Now, we need to go to polar coordinates

$$d^2k_\perp \longrightarrow du\, d\theta$$

$$\begin{cases} d^2k_\perp = (k_\perp)_x (k_\perp)_y \\ (k_\perp)_x = k_\perp \cos\theta \\ \quad = \frac{\pi}{a}\sqrt{u-n^2}\cos\theta \\ (k_\perp)_y = \frac{\pi}{a}\sqrt{u-n^2}\sin\theta \end{cases}$$

Jacobian of transformation gives

$$d^2k_\perp = \begin{pmatrix} \partial k_x/\partial u & \partial k_x/\partial\theta \\ \partial k_y/\partial u & \partial k_y/\partial\theta \end{pmatrix} du\, d\theta$$

$$= \begin{pmatrix} \frac{\pi}{2a}\frac{\cos\theta}{\sqrt{u-n^2}} & -\frac{\pi}{a}\sqrt{u-n^2}\sin\theta \\ \frac{\pi}{2a}\frac{\sin\theta}{\sqrt{u-n^2}} & \frac{\pi}{a}\sqrt{u-n^2}\cos\theta \end{pmatrix}_{\text{determinant}} = \frac{\pi^2}{2a^2}\, du\, d\theta$$

This means,

$$\int d^2k_\perp \mapsto \int \frac{\pi^2}{2a^2}\, du\, d\theta$$

Integrand becomes with changed variable

$$\sqrt{(k_\perp)^2 + \frac{n^2\pi^2}{a^2}} = \sqrt{\frac{\pi^2}{a^2}(u-n^2) + \frac{n^2\pi^2}{a^2}}$$

$$= \frac{\pi}{a}\sqrt{u}$$

Rewriting the answer from Part A in these variables

$$\int d^2k_\perp \sqrt{(k_\perp)^2 + \frac{n^2\pi^2}{a^2}} = \frac{\pi^2}{2a^2}\int_0^{2\pi}\int_0^{\infty} du\,\sqrt{u}\,\frac{\pi}{a}\, f\left(\frac{\pi}{a}\sqrt{u-n^2}\right)$$

$$= \frac{\pi}{2a^2}\,\frac{\pi}{a}\,2\pi \int_{n^2}^{\infty} du\,\sqrt{u}\, f\left(\frac{\pi}{a}\sqrt{u}\right)$$

$$\mathcal{E}_0 = \frac{\hbar c}{8\pi^2}\,\frac{2\pi^4}{2a^3}\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right)\int_{n^2}^{\infty} du\,\sqrt{u}\, f\left(\frac{\pi}{a}\sqrt{u}\right)$$

$$= \frac{\hbar c\pi^2}{8a^3}\left(\cdots - \cdots\right)\int_{n^2}^{\infty} du\,\sqrt{u}\, f\left(\frac{\pi}{a}\sqrt{u}\right)$$

But we see that, we have $\underline{\underline{n^2}}$ ✓, so

$$\sum_{n=-\infty}^{n=\infty} F(n) = 2 \sum_{n=0}^{\infty} \int_{n^2}^{\infty} du \sqrt{u}\, f\left(\frac{\pi}{a}\sqrt{u}\right)$$

$$= 2 \left\{ \frac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n) \right\}$$

We pull this $\underline{\underline{2}}$ out and get

$$\mathcal{E}_0 = \frac{hc\pi^2}{4a^3} \left(\frac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dn\, F(n) \right).$$

Now we use Euler-Mclaurin formula.

$$\frac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dn\, F(n) = -\frac{B_2}{2!} F^{1}(0) - \frac{B_4}{4!} F^{3}(0) + \cdots$$

Also $B_2 = \frac{1}{6}$ and $B_4 = \frac{-1}{30}$.

$$F(n) = \int_{n^2}^{\infty} du \sqrt{u}\, f\left(\frac{\pi}{a}\sqrt{u}\right)$$

Let $\sqrt{u} \rightarrow n$

and using fundamental theorem of calculus gives

$$F'(n) = -2n^2 f\left(\frac{n\pi}{a}\right)$$

$$F'(0) = 0$$

$$F''(n) = -4n \cdot \frac{\pi}{a} f'\left(\frac{n\pi}{a}\right)\Big|_{n=0}$$

$$= 0$$

$$F'''(n) = -4 f'''(0)$$

$$= -4$$ ✓ only third derivative non-vanishing.

using this in $\mathcal{E}_0$ gives

$$\mathcal{E}_0 = \frac{\hbar c \pi^2}{4a^3}\left[-\frac{B_4}{4!} F^3(0)\right]$$

$$= \frac{\hbar c \pi^2}{4a^3} \times \frac{1}{30} \times \frac{1}{4!} \times -4$$

$$= \frac{-\pi^2}{720} \frac{\hbar c}{a^3}$$ ✓

Since $\mathcal{E}_0 = \mathcal{E}_{free} - \mathcal{E}_{boun}$

i.e $(\mathcal{E}_\infty - \mathcal{E}_a)$ → work done $\int_A^B \bar{F} \cdot d\bar{l} = W_A - W_B$

Now $$F_z = -\frac{\partial \mathcal{E}_0}{\partial a} = \frac{+\pi^2}{720} \hbar c \frac{-3}{a^4}$$

$$= \frac{-\pi^2}{240} \frac{\hbar c}{a^4}$$ ✓

when $a = 1\,\mu m$

$= 10^{-6}\, m$

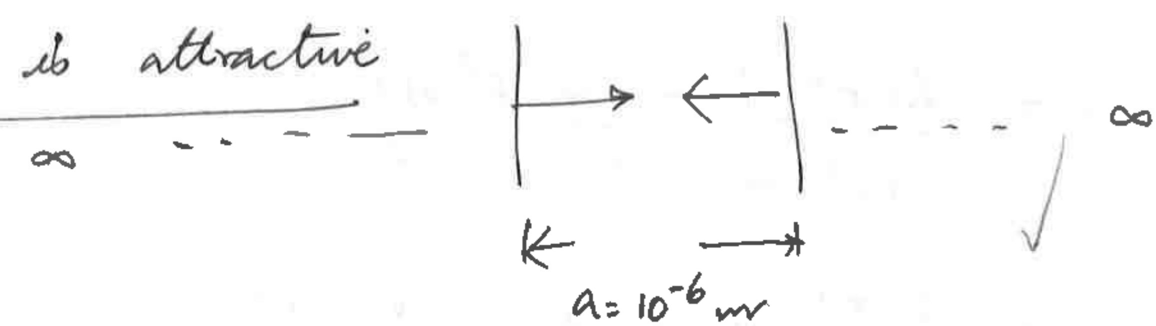
$$\frac{F}{Z} = \frac{-\pi^2}{240} \frac{6.6 \times 10^{-34} \times 3 \times 10^{8}}{(10^{-6})^4}$$

$$= \frac{-\pi^2}{240} \frac{19.8 \times 10^{-20}}{10^{-18}}$$

$$= \frac{-\pi^2}{240} 19.8 \times 10^{-2}$$

$$\cong 10^{-3}\ N/m^2$$

The force is attractive



Problem #2 Srednicki 3.3

Use $U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)$ to show that

$$U(\Lambda)^{-1}a(\bar{k})U(\Lambda) = a(\Lambda^{-1}\bar{k})$$

$$U(\Lambda)^{-1}a^{+}(\bar{k})U(\Lambda) = a^{+}(\Lambda^{-1}\bar{k})$$

Note/caveat :- $U(\Lambda)$ acts on four-vector $\vec{k}$ (Lorentz Transf.).

Define a Fourier transform (FT)

$$\tilde{\phi}(k) = \int d^4x \; e^{-ikx}\varphi(x)$$

and Inv. FT

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\tilde{\phi}(k)$$

✓.

We know that $\varphi(x)$ is hermitian, hence $\tilde{\phi}^{+}(k) = \tilde{\phi}(-k)$

$$U(\Lambda)^{-1}\tilde{\phi}(k)U(\Lambda) = \int d^4x \; e^{-ikx}\,U(\Lambda)^{-1}\tilde{\phi}(k)U(\Lambda)$$

$$= \int d^4x \; e^{-ikx}\,\phi(\Lambda^{-1}x)$$

Set $x = \Lambda y$

s.t $\Lambda^{-1}x = y$

$$= \int d^4y \; e^{-ik\Lambda y}\,\phi(y)$$

$$= \int d^4y \; e^{-ik\Lambda y}\,\phi(y) \quad \cdots \text{①}$$

$$K\Lambda y = k^{\mu}\Lambda_{\mu}{}^{\nu} y_{\nu}$$

$$= \Lambda_{\mu}{}^{\nu} k^{\mu} y_{\nu}$$

$$= (\Lambda^{-1})^{\nu}{}_{\mu} k^{\mu} y_{\nu}$$

$$= (\Lambda^{-1}k)^{\nu} y_{\nu} = (\Lambda^{-1}k)y \quad \cdots \quad (2)$$

using (2) in (1), we get

$$= \int d^4y \, e^{-i(\Lambda^{-1}k)y} \phi(y)$$

$$= \tilde{\phi}(\Lambda^{-1}k)$$

✓.

Now, we would believe the same thing with $a(\bar{k})$ i.e

$$U(\Lambda)^{-1} a(\bar{k}) U(\Lambda) = a(\Lambda^{-1}k)$$

and similarly for $a^{+}(k)$ ✓.

This means that we have to find some relation b/w $\tilde{\phi}(k)$, $a^{+}(\bar{k})$ and $a(\bar{k})$ ...

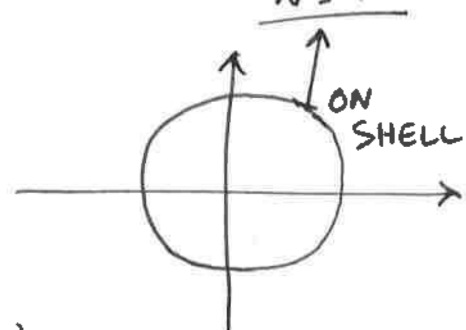
Ansatz :- $\tilde{\phi}(k) = 2\pi\delta(k^2+m^2)\left[\theta(k^0)a(\bar{k}) + \theta(-k^0)a^{+}(-\bar{k})\right]$

I picked this expression of ansatz from an online page www.kitp.ucsb.edu/sites/default/files/users/joep/qft_soln set_2.pdf

If $k^0 > 0$, we obtain

$$2\pi\delta(k^2+m^2)a(\bar{k}) = \tilde{\phi}(k)$$

Physically, this represents ORTHOCHRONOUS LORENTZ TRANSFORMATION strictly ON-SHELL



we get,

$$2\pi\delta(k^2+m^2)\,U(\Lambda)^{-1}a(k)U(\Lambda) = U(\Lambda)^{-1}\tilde{\phi}(k)\,U(\Lambda)$$

$$= \tilde{\phi}(\Lambda^{-1}k) \longrightarrow \text{proved before}$$

$$= 2\pi\delta(((\Lambda^{-1}k)^2+m^2)\,a(\Lambda^{-1}k)$$

$$= 2\pi\delta(k^2+m^2)\,a(\Lambda^{-1}k) \qquad \left\{ (\Lambda^{-1}k)^2 = k^2 \right\} \downarrow \text{invariant under } \Lambda$$

Matching L.H.S & R.H.S, we get

$$U(\Lambda)^{-1}a(\bar{k})\,U(\Lambda) = a(\Lambda^{-1}\bar{k}) \quad \text{with } k^0 > 0$$

$\downarrow$ TAKE CONJUGATE

$UU^\dagger = \mathbb{1}$
$U^\dagger = U^{-1}$

$$U(\Lambda)^{-1}a^\dagger(\bar{k})\,U(\Lambda) = a^\dagger(\Lambda^{-1}k) \quad ✓$$

c) $U(\Lambda)|k_1 \ldots k_n\rangle = |\Lambda k_1, \ldots \Lambda k_n\rangle \quad \ldots\ldots \quad (1)$

use the hint that

$$|k_1 \ldots k_n\rangle = a^\dagger(\bar{k}_1) \ldots\ldots a^\dagger(\bar{k}_n)|0\rangle \qquad \text{in eq. (1)}$$

$$U(\Lambda)\, a^\dagger(\bar{k}_1) \ldots\ldots a^\dagger(\bar{k}_n)\, |0\rangle$$

$$= U(\Lambda) a^\dagger(\bar{k}_1) U(\Lambda)^{-1} \ldots\ldots U(\Lambda) a^\dagger(\bar{k}_n) U(\Lambda)^{-1} U(\Lambda)|0\rangle$$

← bunchs of identities in between

$$= a^\dagger(\Lambda\vec{k}_1) \ldots\ldots a^\dagger(\Lambda\vec{k}_n)\, U(\Lambda)|0\rangle$$

But $U(\Lambda)|0\rangle = |0\rangle$ and we get

$$U(\Lambda) a^\dagger(\bar{k}_1) \ldots\ldots a^\dagger(\bar{k}_n)|0\rangle$$

$$= U(\Lambda)|k_1 \ldots k_n\rangle$$

$$= |\Lambda k_1 \ldots\ldots \Lambda k_n\rangle$$

Proven. ✓

Problem #3 Srednicki 3.5

$$\mathcal{L} = -\partial^\mu \varphi^+ \partial_\mu \varphi - m^2 \varphi^+ \varphi + \Omega_0$$

a) The Euler-Lagrange equation for fields is :

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$$

$$-m^2 \varphi^+ - \partial_\mu \left(-\partial^\mu \varphi^+ \right) = 0$$

$$\Rightarrow \quad (\partial^2 - m^2)\varphi^+ = 0 \quad \checkmark.$$ K-G equation for φ^+

Also, $\frac{\partial \mathcal{L}}{\partial \varphi^+} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^+)} \right) = 0$ gives

$$(\partial^2 - m^2)\varphi = 0 \quad \checkmark .$$

b) $$\Pi^+ = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^+)} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^+} = \dot{\varphi}$$

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}^+$$

Now, constructing Hamiltonian density.

$$\mathcal{H} = \Pi(x)\dot{\varphi}(x) + \Pi^+(x)\dot{\varphi}^+(x) - \mathcal{L}$$
$$= \dot{\varphi}^+ \dot{\varphi} + \dot{\varphi}\dot{\varphi}^+ - \mathcal{L}$$

$$\mathcal{H} = \cancel{\pi(x)\pi^\dagger(x)} + \pi^\dagger(x)\pi(x) - \cancel{\pi(x)\pi^\dagger(x)} + \nabla\phi^\dagger\nabla\phi + m^2\varphi^\dagger\varphi - \Omega_0$$

$$= \pi^\dagger(x)\pi(x) + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi - \Omega_0$$

✓

c) Write the mode expansion of φ as:

$$\varphi(x) = \int d\tilde{k} \left[a(\bar{k})e^{ikx} + b^\dagger(\bar{k})e^{-ikx}\right]$$

We follow Srednicki p-26 for this derivation

$$\int d^3x \, e^{-ikx}\varphi(x) = \frac{1}{2\omega}a(\bar{k}) + \frac{1}{2\omega}e^{2i\omega t}b^*(-\bar{k}) \quad \ldots (1)$$

Now do the same thing but $\varphi(x) \longmapsto \partial_0\varphi(x)$, we get.

$$\int d^3x \, e^{-ikx}\partial_0\varphi(x) = \frac{-i}{2}a(\bar{k}) + \frac{i}{2}e^{2i\omega t}b^\dagger(-\bar{k}) \quad \ldots (2)$$

we can combine (1) & (2) to get

$$a(\bar{k}) = \int d^3x \, e^{-ikx}\left[i\partial_0\varphi(x) + \omega\varphi(x)\right]$$

$$= i\int d^3x \, e^{-ikx}\overleftrightarrow{\partial_0}\varphi(x)$$

✓

Also, one thing to note is

$$\varphi(x) \longmapsto \varphi^\dagger(x) \text{ changes } a(\bar{k}) \longmapsto b(\bar{k})$$

Therefore,

$$b(\bar{k}) = i \int d^3x \, e^{-ikx} \overleftrightarrow{\partial_0} \varphi^+(\bar{x})$$

d) $$a(\bar{k}) = \int d^3x \, e^{-ikx}\left(\omega\varphi(x) + i\,\partial_0\varphi(x)\right)$$

But $\partial_0\varphi(x) = \pi^+(x)$

$$a(\bar{k}) = \int d^3x \, e^{-ikx}\left(\omega\varphi(x) + i\,\pi^+(x)\right) \quad \ldots (1)$$

And,

$$b(\bar{k}) = \int d^3x \, e^{-ikx}\left(\omega\varphi^+(x) + i\,\pi(x)\right) \quad \ldots (2)$$

clearly $[a(\bar{k}), a(\bar{k}')] = 0$ as is obvious from (2)

Also $[a^+(\bar{k}), a^+(\bar{k}')] = 0$

Similarly for $[b(\bar{k}), b(\bar{k}')] = 0 = [b^+(\bar{k}), b^+(\bar{k}')]$

Also $[a, b^+]$ and $[a^+, b]$ vanishes since we can always change $\varphi(x) \mapsto \varphi^+(x)$ and that changes $a(\bar{k}) \mapsto b(\bar{k})$

Only non-vanishing commutators are $[a, a^+]$ and $[b, b^+]$

$$[a, a^\dagger] = \left[\int d^3x\, e^{-ikx} \left(\omega\varphi(x) + i\Pi^\dagger(x)\right), \int d^3x'\, e^{ik'x'}\left(\omega\varphi^\dagger(x') - i\Pi(x')\right)\right]$$

$$= \int d^3x\, d^3x'\, e^{-i(k-k')x} \left[\omega\varphi(x) + i\pi^\dagger(x),\ \omega\varphi^\dagger(x') - i\pi(x')\right]$$

$$= \int d^3x\, d^3x'\, e^{-i(k-k')x} \left\{ -i\omega[\varphi(x), \pi(x')] - i\omega[\varphi^\dagger(x'), \pi^\dagger(x)] \right\}$$

use Eq. 3.28 from Srednicki i.e

$$[\varphi(x), \pi(x')] = i\delta^3(x-x')$$

$$[\varphi^\dagger(x), \pi^\dagger(x')] = -i\delta^3(x-x')$$

we get,

$$[a, a^\dagger] = \int d^3x\, d^3x'\, e^{-i(k-k')x} \left\{ -i\omega(i\delta^3(x-x')) + i\omega(-i\delta^3(x-x')) \right\}$$

$$= 2\omega \int d^3x\, e^{-i(k-k')x}$$

$$= (2\pi)^3 \delta^3(\bar{k}-\bar{k}')\, 2\omega \quad \checkmark$$

For $[b, b^\dagger]$ we can change

$$\Pi \longleftrightarrow \Pi^\dagger$$

$$\phi \longleftrightarrow \phi^\dagger$$ and get

$$[b, b^\dagger] = (2\pi)^3\, 2\omega\, \delta^3(\bar{k} - \bar{k}')$$ ✓✓

e) The expression for $\underline{H}$ in terms of $a^\dagger(k), a(k), b^\dagger(k)$ & $b(k)$ is given by: (↳ for K-G Lagrangian)

$$H = \omega \int d\tilde{k} \left(a^\dagger(k)a(k) + b(k)b^\dagger(k)\right)$$

Ref :- Any standard QFT text.

Now since in our case, there is additional source Ω_0, hence

$$H = -\Omega_0 V + \omega \int d\tilde{k} \left(a^\dagger(k)a(k) + b(k)\, b^\dagger(k)\right) \quad \cdots ②$$

Now we know

$$[b, b^\dagger] = (2\pi)^3\, 2\omega\, \delta^3(\bar{k} - \bar{k}') \quad \cdots ③$$

i.e

$$b(k)\, b^\dagger(\bar{k}') = b^\dagger(\bar{k}')\, b(\bar{k}) + (2\pi)^3\, 2\omega\, \delta^3(\bar{k} - \bar{k}') \quad \cdots ④$$

using ④ in ② we get.

$$H = -\Omega_0 V + \omega \left\{ \int d\tilde{k} \left(a^\dagger(k)a(k) + b^\dagger(k)\, b(k) + (2\pi)^3\, 2\omega\, \delta^3(0)\right\}\right.$$

use $(2\pi)^3\, \delta^3(0)$ as volume (As in Srednicki) P-28,

$$= -\Omega_0 V + \omega \left\{ \int d\tilde{k} \left(a^\dagger(k)a(k) + b^\dagger(k)\, b(k) + 2\omega V\right\}\right. \quad (*)$$

Also $\mathcal{E}_0 = \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3k\, \omega$ & $d\tilde{k} = \frac{d^3k}{(2\pi)^3}\, 2\omega \frac{1}{2\omega}$

we get $H = (2\mathcal{E}_0 - \Omega_0) V + \int d\tilde{k}\; \omega \left[a^\dagger(k)a(k) + b^\dagger(k)b(k)\right]$ ✓

In order for ground state to have zero energy we must choose $2\mathcal{E}_0 - \Omega_0 = 0$

$$\Rightarrow \boxed{\Omega_0 = 2\mathcal{E}_0}$$ ✓.

Problem #4 Srednicki 6.1

$$\langle q'', t'' | q', t' \rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t}$$

... 6.7 Srednicki

We' are to simplify this.

let us call this ξ.

$$\xi = \prod_{j=0}^{N} \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t} \quad \cdots \quad (1)$$

Let $p_j(q_{j+1}-q_j) := p_j a_j := p_j \dot{q}_j \delta t$ ... (2)

using (2) in (1)

$$\xi = \prod_{j=0}^{N} \int \frac{dp_j}{2\pi} e^{ip_j a_j} e^{\frac{-ip_j^2 \delta t}{2m}}$$

using the standard Gaussian integral $\int e^{-\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha}$

we get,

$$\xi = \prod_{j=0}^{N} \frac{1}{2\pi} \sqrt{\frac{\pi . 2m}{i\delta t}} e^{-a_j^2/4i\delta t \, . \, 2m}$$

$$= \prod_{j=0}^{N} \sqrt{\frac{m}{2\pi i \delta t}} \cdot \exp\left[\left(i\delta t \sum_j \frac{1}{2} m \dot{q}_j^2\right)\right]$$

$$= \left(\frac{m}{2\pi i \delta t}\right)^{\frac{N+1}{2}} \exp\left[i\delta t \sum_j \frac{1}{2} m \dot{q}_j^2\right]$$

$$\mathcal{D}q = C \prod_{j=1}^{N} dq_j$$

and $C = \left(\frac{m}{2\pi i \delta t}\right)^{\frac{N+1}{2}}$ ✓

b) $$\langle q''|t''|q',t'\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-iH(p_j,\bar{q}_j)\delta t}$$

Now, use the result from Part (a)

$$= \left(\frac{m}{2\pi i \delta t}\right)^{\frac{N+1}{2}} \int \prod_{k=1}^{N} dq_k \exp\left[\frac{im}{2}\delta t \left(\sum_{j=0}^{N} \frac{(q_{j+1}-q_j)^2}{(\delta t)^2}\right)\right]$$

let $\frac{i\delta t}{m} = A$

$$= \left(\frac{1}{2\pi A}\right)^{\frac{N+1}{2}} \int \prod_{k=1}^{N} dq_k \exp\left[\frac{-m}{2i\delta t}\left(\sum_{j=0}^{N}(q_{j+1}-q_j)^2\right)\right]$$

$$= \left(\frac{1}{2\pi A}\right)^{\frac{N+1}{2}} \int \prod_{k=1}^{N} dq_k \exp\left[\frac{-1}{2A}\left(\sum_{j=0}^{N}(q_{j+1}-q_j)^2\right)\right]$$

The integral over q_1 is

$$\int dq_1 \exp\left(-\frac{(q_2-q_1)^2}{2A}\right)\exp\left(-\frac{(q_1-q_0)^2}{2A}\right)$$

$$\int dq_1\, e^{\frac{-1}{2A}\left[q_2^2+q_1^2-2q_1q_2+q_1^2+q_0^2-2q_1q_0\right]}$$

$$\int dq_1\, e^{\frac{-1}{2A}\left[2q_1^2-2q_1(q_2+q_0)+(q_0^2+q_2^2)\right]}$$

$$\int dq_1\, e^{\frac{-1}{A}\left[q_1^2-q_1(q_2+q_0)+\left(\frac{q_0^2+q_2^2}{2}\right)\right]}$$

$$= \sqrt{\frac{2A\pi}{2}} \; e^{-\frac{(q_2-q_0)^2}{4A}}$$

The integral over 'q_2' is now

$$\int dq_2 \; \exp\left(-\frac{(q_3-q_2)^2}{2A}\right) \exp\left(-\frac{(q_2-q_0)^2}{4A}\right)$$

gives us $\sqrt{\frac{2}{3}(2A\pi)} \; \exp\left(\frac{-(q_3-q_0)^2}{6A}\right)$ **

We can now follow the result of integration over dq_N is

$$\int dq_N \, e^{-(q_{N+1}-q_N)^2/2A} \, e^{-\frac{(q_N-q_0)^2}{2AN}} = \sqrt{\frac{N}{N+1}(2A\pi)} \, e^{-\frac{(q_{N+1}-q_0)^2}{2(N+1)A}}$$

Hence,

$$\prod_{k=1}^{N} dq_k \; \exp\left(\frac{-1}{2A}\sum_{j=0}^{N} (q_{j+1}-q_j)^2\right) = \sqrt{\frac{1}{N+1}} \; (2\pi A)^{N/2} \, e^{-\frac{(q_{N+1}-q_0)^2}{2(N+1)A}}$$

So,

$$\langle q'', t'' | q', t' \rangle = \left(\frac{1}{2\pi A}\right)^{\frac{N+1}{2}} \frac{1}{(N+1)^{1/2}} (2\pi A)^{N/2} \; e^{-\frac{(q_{N+1}-q_0)^2}{2(N+1)A}}$$

$$= \sqrt{\frac{1}{2\pi A(N+1)}} \; e^{-\frac{(q_{N+1}-q_0)^2}{2(N+1)A}}$$

$$= \sqrt{\frac{m}{2\pi i \delta t (N+1)}} \; e^{-\frac{(q_{N+1}-q_0)^2 m}{2(N+1) i \delta t}}$$

$$= \sqrt{\frac{m}{2\pi i (t''-t)}} \; e^{+\frac{i(q_{N+1}-q_0)^2 m}{2(t''-t)}}$$ ✓

Now, $\underline{L\,dt}$ has units of $\underline{\hbar}$. So to fix the dimension; we will have

$$\langle q'', t'' | q', t' \rangle = \sqrt{\frac{m}{2\pi i\hbar (t''-t')}}\; e^{\frac{im(q''-q)^2}{2\hbar (t''-t)}}$$

(#) ✓

c) $$\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-\frac{i\hat{p}^2}{2m}(t''-t)} | q' \rangle$$

$$= \int_{-\infty}^{\infty} dp\, \langle q'' | e^{-\frac{i\hat{p}^2}{2m}(t''-t')} | p \rangle \langle p | q' \rangle$$

$$= \int_{-\infty}^{\infty} dp\, \langle q'' | p \rangle \langle p | q' \rangle\, e^{-i\frac{p^2}{2m}T}$$

$\underline{T = t'' - t'}$

$$= \int_{-\infty}^{\infty} dp\, \frac{e^{ipq''}}{\sqrt{2\pi}} \frac{e^{-ipq'}}{\sqrt{2\pi}}\, e^{-i\frac{p^2}{2m}T}$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ip(q''-q') - i\left(\frac{p^2}{2m}\right)T}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2m\pi}{iT}}\; e^{-\frac{(q''-q')^2\cdot 2m}{4iT}}$$

$$= \sqrt{\frac{m}{2\pi i T}}\; e^{\frac{im(q''-q')^2}{2T\hbar}}$$

(#) ✓

Clearly (#) and (#) agree ✓

Problem #5 Srednicki 8.4

$$\phi(x) = \int d\tilde{k}\left[a(\vec{k})e^{ikx} + a^*(\vec{k})e^{-ikx}\right] \quad \cdots \quad 3.19$$

$$\left[a(\vec{k}),\, a^\dagger(\vec{k}')\right] = (2\pi)^3\, 2\omega\, \delta^3(\vec{k}-\vec{k}') \quad \cdots \quad 3.29$$

$$a(\vec{k})|0\rangle = 0 \quad \cdots \quad 5.3$$

Eq. 3.19 implies:

$$\langle 0|T(\phi(x_1)\phi(x_2))|0\rangle = \langle 0|\, T\int d\tilde{k}_1\, d\tilde{k}_2\left[a(\vec{k}_1)e^{ik_1x_1} + a^\dagger(\vec{k}_1)e^{-ik_1x_1}\right]\left[a(\vec{k}_2)e^{ik_2x_2} + a^\dagger(\vec{k}_2)e^{-ik_2x_2}\right]|0\rangle$$

for $t_1 > t_2$

$$= \langle 0|\, T\int d\tilde{k}_1\, d\tilde{k}_2\left[a(\vec{k}_1)e^{ik_1x_1}\, a^\dagger(\vec{k}_2)e^{-ik_2x_2}\right]|0\rangle$$

→ using 5.3

T → saves causality
→ time ordering → assume it is there

$$= \langle 0|\int d\tilde{k}_1\, d\tilde{k}_2\left[a(\vec{k}_1)e^{ik_1x_1}\, a^\dagger(\vec{k}_2)e^{-ik_2x_2}\right]|0\rangle$$

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \int d\tilde{k}_1\, d\tilde{k}_2\, e^{ik_1x_1 - ik_2x_2}\langle 0|a(\vec{k}_1)a^\dagger(\vec{k}_2)|0\rangle$$

$$= \int d\tilde{k}_1\, d\tilde{k}_2\, e^{ik_1x_1 - ik_2x_2}(2\pi)^3\, 2\omega_2\, \delta^3(\vec{k}_1-\vec{k}_2) \quad \cdots \quad (5)$$

Note: we have used Eq. 3.29 and the fact that since $t_1 > t_2$, particle is created at t_2 and annihilated at t_1.

using $d\tilde{k}_2 = \dfrac{d^3k_2}{(2\pi)^3\, 2\omega_2}$ in (5), we get

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = \int d\tilde{k}_1\, d^3k_2\, e^{ik_1x_1 - ik_2x_2}\, \delta^3(\bar{k}_1 - \bar{k}_2)$$

$$= \int d\tilde{k}_1\, e^{ik_1(x_1-x_2)}$$

Now, if $t_2 > t_1$, we have to flip t_1 & t_2

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = \Theta(t_1-t_2)\int d\tilde{k}\, e^{ik(x_1-x_2)} + \Theta(t_2-t_1)\int d\tilde{k}\, e^{-ik(x_1-x_2)} \quad \ldots (6)$$

Now, from Srednicki Eq. 8.13, we have

$$\Delta(x-x') = i\Theta(t-t')\int d\tilde{k}\, e^{ik(x-x')} + i\Theta(t'-t)\int d\tilde{k}\, e^{-ik(x'-x)} \quad \ldots (7)$$

using (7), (6) becomes

$$= \frac{1}{i}\Delta(x_2 - x_1)$$

✓ which was required

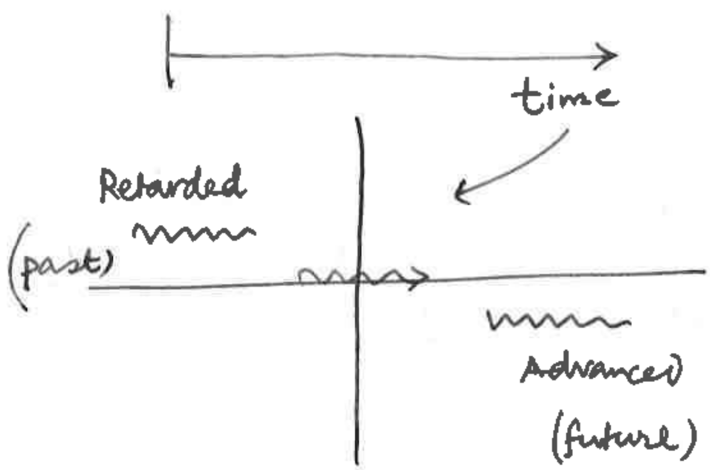
Problem # 6 Srednicki 8.5

Retarded Green $f^{\underline{n}}$ $\Delta_{ret}(x-y) = 0$ for $x^0 \geq y^0$

Advanced Green $f^{\underline{n}}$ $\Delta_{ad}(x-y) = 0$ for $x^0 \leq y^0$

Find pole prescription on RHS of 8.11

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon}$$



Pole Structure of Green's function.

For $x^0 \geq y^0$ i.e $\Delta_{retarded}(x-y) = 0$, we must choose contour in the lower half k^0 plane. And the result will vanish (we want that) if the poles are above the real axis.

$$\Delta_{ret}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{-(k^0)^2 + \vec{k}^2 + m^2 - i\epsilon}$$

Below the k^0 plane, we change by $i\epsilon$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{-(k^0 - i\epsilon)^2 + \vec{k}^2 + m^2}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 + 2ik^0\epsilon}$$

k^2 ↓ four vector

$$= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 + i(k^0)\epsilon'}$$

{ Let $2\epsilon = \epsilon'$, ϵ is small

But $k^2 > 0$

$m^2 > 0$

Sign of $i(k^0)\epsilon$ determines poles

$$= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 + i\overleftrightarrow{k_0}\epsilon'}$$

$\overleftrightarrow{k_0}$ = Sign k_0 + or −

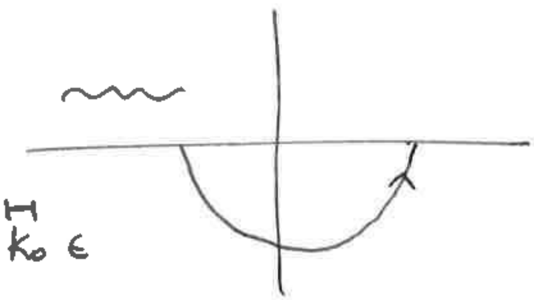
✓

⇓

Retarded Green f^n.

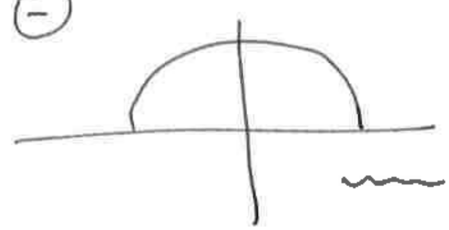
For retarded, we integrate over upper plane in order to skirt poles lying below and get an answer of 0 since no residues included.

$$\Delta_{ret}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2+m^2 - i\,\overset{\text{H}}{k_0}\,\epsilon}$$



$\overset{\text{H}}{k_0}$ = Sign of k_0

⊕ or ⊖



In these situations, the Green's functions (advanced & retarded) will vanish.

Q.F.T Assignment 5

K. JHA

02/25/2014

59/60

Problem # 1 Srednicki 9.2

$$Z(J) = \langle 0|0\rangle_J$$

$$= \int \mathcal{D}\phi \, e^{i\int d^4x [\mathcal{L}_0 + \mathcal{L}_I + J\phi]}$$

i.e $$Z(J) = e^{i\int d^4x \, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} \int \mathcal{D}\phi \, e^{i\int d^4x [\mathcal{L}_0 + J\phi]}$$

$$\propto e^{i\int d^4x \, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)} Z_0(J)$$ — Srednicki 9.6

Now here, $\mathcal{L}_1 = -\frac{1}{24} Z_\lambda \lambda \phi^4 + \mathcal{L}_{ct}$

$$Z_1(J) \propto \exp\left[\frac{-i}{24} Z_\lambda \lambda \int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right] Z_0(J)$$

$$Z_1(J) \propto \sum_{V=0}^{\infty} \frac{1}{V!}\left[\frac{-i}{24} Z_\lambda \lambda \int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right]^V \cdot \sum_{P=0}^{\infty} \frac{1}{P!}\left[\frac{i}{2}\int d^4y \, d^4z \, J(y)\Delta(y-z) J(z)\right]^P$$

No. of surviving sources is $E = 2P - 4V$ ✓

Overall phase factor $= (-i)^V \left(\frac{1}{i}\right)^{4V} i^P = (i)^{-V} (i)^{-4V} (i)^P$

$$= (i)^{-V} (i)^{E-P}$$

$$= (i)^{E-P-V}$$ ✓

The vertex joins <u>4</u> line segments.

Associated V. Factor = $4! \frac{-i}{24} \lambda \int d^4x$ { 4! since we have 24 ways of arranging vertex legs.

$= -i\lambda \int d^4x$ ✓.

$\frac{1}{i} \Delta(x-y)$

Propagator

$i \int d^4x \, J(x)$

Source

b) $1 \leq E \leq 4$
$0 \leq V \leq 2$ } Constraint $E = 2P - 4V$.

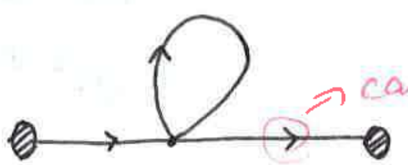
Odd - E won't work i.e $E \neq 1$, $E \neq 3$

$E = 2$ $V = 0$ $P = 1$

$S = 2$

S is the symmetry factor.

$E = 2$ $V = 1$ $P = 3$



careful → arrows normally indicate charge flow

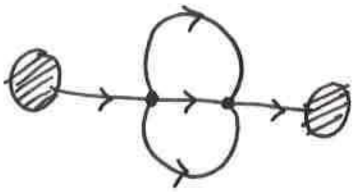
$S = 2 \times 2 = 4$

(Exchange ends of internal propagator & sources).

E = 2

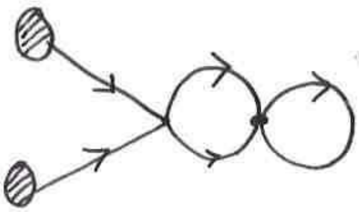
V = 2

P = 5



$S = 2 \times 3!$
$= 12$ ✓

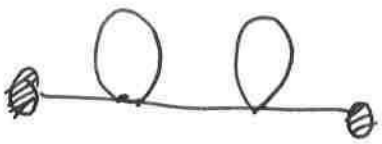
[3! ways to arrange propagators and 2 to flip them].



$S = 2^3$
$= 8$ ✓

[Since, this is same as E = 2, V = 1 diagram but here we can also exchange propagator!]

$2!\,2!\,2! = \underline{8}$

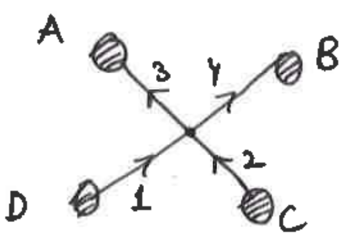


$S = 2^3$
$= 8$ ✓

E = 4

V = 1

P = 4



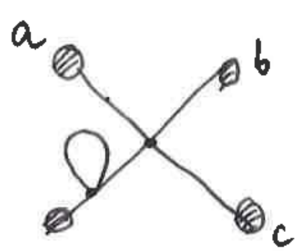
$S = 4!$ ✓

[we can write ABCD & 1234]
and then check <u>24</u> <u>possibilities</u>

E = 4

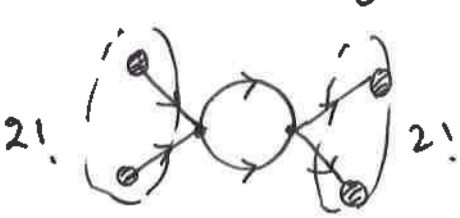
V = 2

P = 6



$S = 3! \times 2$
$= 12$ ✓

[abc in 3! ways and fourth one in 2 ways]



$S = 2! \times 2! \times 2 \times 2$
$= 24$ ✓

is Lagrangian has $\phi \to -\phi$ symmetry. Here,
e have no value of $E=1$. Tadpoles don't
rise. Why cancel them?

$E=1$

Not present here

Problem # 2 Srednicki - 9.3

Since, in this case we have φ^+ and φ. we have same argument as last problem just that 4! ways is now $(2!)(2!) = 4$

Vertex Factor $= \cancel{2!}\,\cancel{2!}\; i\int d^4x \; -\frac{1}{\cancel{4}}\lambda$

$= -i\lambda\int d^4x$

By convention:

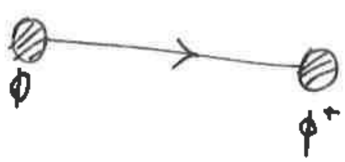
Arrow pointing away is J
" " toward is J^+

} Clearly 'S' will be less than previous problem. Since, they are not interchangeable.

E = 2

V = 0

P = 1

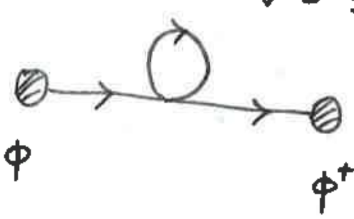


S = 1

E = 2

V = 1

P = 3

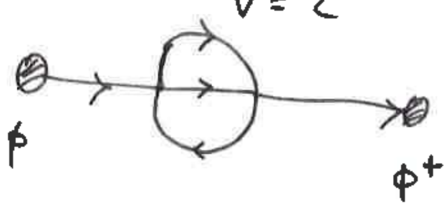


S = 1

E = 2

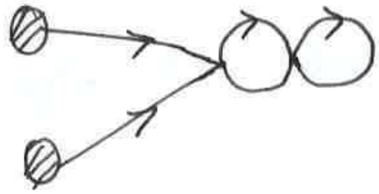
V = 2

P = 5



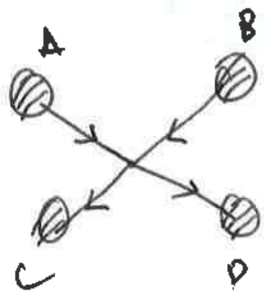
S = ~~1/2~~ S = 2 ! → Same arrow signs can be flipped.

$S = 1$ ✓



$S = 1$ ✓

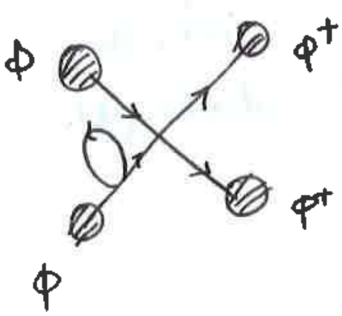
$E = 4$ $V = 1$ $P = 4$



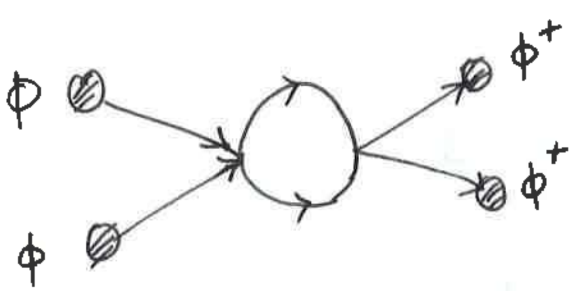
$S = 2 \times 2$
$= 4$ ✓

flip AB and CD together (holding other fixed).

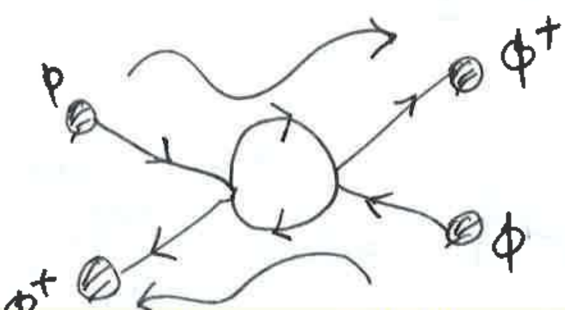
$E = 4$ $V = 2$ $P = 6$



$S = 2$ ✓



$S = 2 \times 2 \times 2$
✓ $= 8$ [ϕ's can be flipped → 2
ϕ^+'s can be flipped → 2
◯ can be flipped → 2



$S = 2$ ✓ Arrows opposite, can't do flipping.

Problem #3 Srednicki 9.5

a) $\phi(x,t) \equiv e^{iHt} \varphi(x,0) e^{-iHt}$

Define

$$\phi_I(x,t) \equiv e^{iH_0 t} \varphi(x,0) e^{-iH_0 t} \quad — (2)$$

The BCH formula reads,
(Lemma actuly)

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[\;---\;]$$

we have from (2)

$$\Phi_I = \Phi_S + ti[H_0, \phi_S] + \frac{i^2 t^2}{2!}[H_0,[H_0,\phi_S]] + --- \quad — (3)$$

But we have $H_0 = \int \mathcal{H}_0 \, d^3x$

$$= \int \left(\frac{1}{2}\Pi^2 + \frac{(\nabla\phi)^2}{2} + \frac{m^2\phi^2}{2}\right) d^3x$$

$$= \int d\tilde{k}\, \omega a^\dagger(\bar{k}) a(\bar{k}) \quad \text{where } \omega = \sqrt{k^2+m^2}$$

$$[H_0, a(k)] = \int d\tilde{k}' \, \omega(k') [a^\dagger(k')a(k'), a(k)]$$

$$= -\omega(k) a(k)$$

$$[H_0, a^\dagger(k)] = \omega(k) a^\dagger(k)$$

using this

$[AB,C] = A[B,C] + [A,C]B$

$[a^\dagger(k')a(k'), a(k)] = (2\pi)^3 2\omega \, \delta^3(k'-k)$

Now $[H_0, \phi_S] = \int d\tilde{k} \, [H_0, a(k)e^{ikx} + a^\dagger(k)e^{-ikx}]$

$$= \int d\tilde{k} \, (-\omega a(k) e^{ikx} + \omega a^\dagger(k) e^{-ikx}] \quad — (4)$$

Using ④ in ③, we get (and seeing the trend)

$$\phi_I = \int d\tilde{k} \left[e^{ik\cdot x} e^{-i\omega t} a(k) + e^{-ik\cdot x} e^{i\omega t} a^+(k) \right]$$

Now take double time & Space derivative of Φ_I

$$\frac{\partial^2}{\partial t^2} \Phi_I = -\omega^2 [\quad \Phi_I \quad]$$

$$\frac{\partial^2}{\partial x^2} \Phi_I = -k^2 [\quad \Phi_I \quad]$$

But, here signature is $- + + +$

So $(\omega^2 - k^2) \Phi_I = -\Box^2 \Phi_I$

where $\Box^2 = \frac{-\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$ or just ∂^2

$$\Rightarrow \left(-\partial^2 + m^2\right) \Phi_I = 0$$

This is K-G equation ✓.

b) Heisenberg picture field is

$$\varphi(\bar{x}, t) = e^{iHt} \varphi(x,0) e^{-iHt} \quad \ldots \quad ①$$

and Interaction picture is ... ②

$$\varphi_I(\vec{x}, t) = e^{iH_0 t} \varphi(x,0) e^{-iH_0 t}$$

$$\Rightarrow \varphi(x,0) = e^{-iH_0 t} \varphi_I(\bar{x}, t) e^{iH_0 t} \quad \ldots \quad ③$$

Insert (3) in (1) we get,

$$\varphi(\vec{x},t) = e^{iHt} e^{-iH_0 t} \Phi_I(x,t) e^{iH_0 t} e^{-iHt}$$

$$= U^{\dagger}(t) \Phi_I(x,t) U(t)$$

c) $i \frac{d}{dt} U(t) = H_I(t) U(t)$ .... (1)

$$\text{L.H.S} = i \frac{d}{dt} U(t)$$

$$\Rightarrow i \frac{d}{dt} e^{iH_0 t} e^{-iHt}$$

$$= i \left[iH_0 e^{iH_0 t} e^{-iHt} + e^{iH_0 t} (-iH) e^{-iHt} \right]$$

$$= e^{iH_0 t} (H) e^{-iHt} - (H_0) e^{iH_0 t} e^{-iHt}$$

$$= e^{iH_0 t} H_1 e^{-iHt}$$

$$= e^{iH_0 t} H_1 \underbrace{e^{-iH_0 t} e^{iH_0 t}}_{\mathbb{1}} e^{-iHt} \quad [\text{insert}$$

$$= H_I(t) U(t)$$

d) Let $\mathcal{H}_I$ be expanded as below

$$\mathcal{H}_I = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} C_{ab} [\Pi(x,0)]^a [\varphi(x,0)]^b$$

Time evolution i.e $\mathcal{H}_I(t)$

$$\mathcal{H}_I(t) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} C_{ab}\, e^{iH_0 t} [\Pi(x,0)]^a [\phi(x,0)]^b e^{-iH_0 t}$$

$\mathbb{1}$ (insert bunch of identities. In fact, i+j identities

$$= \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} C_{ab} [\Pi(x,t)]^a [\phi(x,t)]^b$$

Same f^n of Π & ϕ

✓

e) Show that for $t > 0$

$$U(t) = T \exp\left[-i \int_0^t dt' \, H_I(t')\right]$$

Satisfies $i \frac{dU}{dt} = HU$

$$U(t) = T - iT \int_0^t dt_1 H(t_1) + \sum_{n=2}^{\infty} (-i)^n T \int_0^t H(t_1) dt_1 \cdots \int_0^{t_{n-1}}$$

breaking up the exponential

Now we see that

$$\frac{d}{dt}\left[T\text{-}iT\int_0^t dt_1 H_I(t_1)\int_0^{t_1}\cdots\int_0^{t_{n-1}} dt_n\, H_I(t_n)\right]$$

$$= H_I(t)\int_0^{t_*} dt_2\, H(t_2)\cdots\int_0^{t_{n-1}} dt_n\, H_I(t_n)$$

by fundamental theorem of calculus.

Now switch $t_{i+1} \to t_i$, we get

$$\frac{d}{dt}[\ldots] = T H(t)\int_0^t dt_1 H_I(t_1)\cdots\int_0^{t_{n-2}} dt_{n-1}\, H_I(t_{n-1})$$

$$= 0 + TH_I(t) + T\sum_{n=2}^{\infty}\cdots(-i)^{n-1} H_I(t)\int_0^t dt_1 H(t_1)\cdots\int_0^{t_{n-2}} dt_{n-1} H(t_{n-1})$$

Now let $n \to n+1$

$$i\frac{dU}{dt} = T H_I(t)\left[\mathbb{1} + \sum_{n=1}^{\infty}(-i)^n\int_0^t dt_1 H(t_0)\cdots\int_{t_0}^{t_{n-1}} dt_n\, H(t_n)\right.$$

$$= H_I(t)\left[T + \sum_{n=1}^{\infty} T(-i)^n\int_0^t dt_1\, H(t_0)\cdots\int_0^{t_{n-1}} dt_n\, H(t_n)\right]$$

$$= H_I(t)\, U(t)$$

For $t<0$, we need to replace the Time-ordering by anti-time ordering operator. Flipping everything again.

f) we have

$$U(t_2) = T \exp\left[-i \int_0^{t_2} dt' \, H_I(t')\right]$$

$$U(t_1) = T \exp\left[-i \int_0^{t_1} dt' \, H_I(t')\right]$$

coming from dagger.

$$U(t_2, t_1) = T \exp\left[-i \int_0^{t_2} dt' \, H_I(t')\right] \bar{T} \exp\left[i \int_0^{t_1} dt' \, H_I(t')\right]$$

Since $t_2 > t_1$, break at t_1

$$= T \exp\left[-i \int_{t_1}^{t_2} dt' \, H_I(t')\right] \underline{\exp\left[-i \int_0^{t_1} dt' \, H_I(t')\right]}$$

$$\underline{\bar{T} \exp\left[+i \int_0^{t_1} dt' \, H_I(t')\right]}$$

The underline terms cancel out and give zero.

we get,

$$T \exp\left[-i \int_{t_1}^{t_2} dt' \, H_I(t')\right] = U(t_2, t_1)$$

For $t_1 > t_2$, we do same steps; just that we get $\bar{T}$ in place of T

i.e Anti-time ordering in place of time-ordering

g) From last part, we have

$$U(t_2, t_1) = U(t_2) U^{\dagger}(t_1)$$

we have, $t_2 \to t_1$

$$U(t_1, t_2) = U(t_1) U^+(t_2)$$

Taking dagger, we get

$$U^+(t_1, t_2) = U(t_2) U^+(t_1) \quad \Big\} \text{ by definition}$$
$$= U(t_2, t_1)$$

Again $U(t_3, t_1) = U(t_3) U^+(t_1)$

$$= U(t_3) \underbrace{U^+(t_2) U(t_2)}_{\mathbb{1}} U^+(t_1) \quad \text{inserting identity}$$

$$= U(t_3, t_2) U(t_2, t_1)$$

h) Show that

$$\varphi(x_n) \ldots \varphi(x_1) = U^+(t_n, 0) \varphi_I(x_n) U(t_n, t_{n-1}) \ldots U(t_2, t_1) \phi_I(x_1) U(t_1, 0)$$

R.H.S $= U^+(t_n, 0) \varphi_I(x_n) U(t_n, t_{n-1}) \ldots U(t_2, t_1) \phi_I(x_1) U(t, 0)$

But from part (g) we had $U^+(t_1, t_2) = U(t_2, t_1)$

$\Rightarrow U(0, t_n) \Phi_I(x_N) U(t_n, t_{n-1}) \ldots U(t_2, t_1) \phi_I(x_1) U(t, 0)$

But $U(0) = 0$ i.e $U(0, t_n) = U^+(t_n)$

$\Rightarrow U^+(t_n) \Phi_I(x_N) U(t_n) U^+(t_{n-1}) \ldots U(t_2) U^+(t_1) \phi_I(x) U(t, 0)$

From part (b) this is just the L.H.S ✓

i) In part g, we showed that

$$U(t_3,t_1) = U(t_3,t_2)\,U(t_2,t_1) \quad — \quad (1)$$

and $U^\dagger(t_1,t_2) = U(t_2,t_1) = U(t_2)U^\dagger(t_1) \quad - \quad (2)$

let $t_1 = t_n$, $t_2 = 0$

Eq (1) $\Rightarrow$

$U^\dagger(t_3,t_2)U(t_3,t_1) = U(t_2,t_1)$

$U^\dagger(t_3,t_1)U(t_3,t_2) = U^\dagger(t_2,t_1)$

$$U^\dagger(t_n,0) = U^\dagger(\infty,0)\,U(\infty,t_n)$$

Similarly, other follows. ✓

j) $$U(-\infty,0) = e^{i(1-i\epsilon)H_0(-\infty)}\,e^{-iH(-\infty)}$$

$$U(-\infty,0)|0\rangle = e^{i(1-i\epsilon)H_0(-\infty)}\,e^{-iH(-\infty)}|0\rangle$$

gives $|0\rangle$

$$= e^{-i(1-i\epsilon)H_0(\infty)}\,|0\rangle$$

Expanding as $\sum_n |n\rangle\langle n|0\rangle$

$$= \sum_n e^{-i(1-i\epsilon)H_0(\infty)}\,|n\rangle\langle n|0\rangle$$

Now, all values of $n = 0 \ldots \infty$ are possible, but $e^{-i\infty}|n\rangle = 0$ $\forall\, n \neq 0$

$$= e^{-i(1-i\epsilon)H_0(\infty)}\,|\phi\rangle\langle\phi|0\rangle$$

$|\phi\rangle$ is the vacuum state

$$= |\phi\rangle\langle\phi|0\rangle$$

Taking a dagger (adjoint) of above, we get

$$\langle 0|U^\dagger(\infty,0) = \langle 0|\phi\rangle\langle\phi|$$ ✓

k) Part – k, we proved

$$\varphi(x_n)\cdots\varphi(x_1) = U^\dagger(t_n,0)\,\phi_I(x_n)\,U(t_n,t_{n-1})\,\phi_I(x_{n-1}) \cdots U(t_2,t_1)\,\varphi_I(x_1)\,U(t_1,0)$$

Sandwich it between $\langle 0|\cdots|0\rangle$

$$\langle 0|\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle 0|U^\dagger(t_n,0)\phi_I(x_n)U(t_n,t_{n-1})\,\phi_I(x_{n-1})\cdots U(t_2,t_1)\phi_I(x_1)U(t_1,0)|0\rangle \quad \cdots (2)$$

Now use part i

$$U^\dagger(t_n,0) = U^\dagger(\infty,0)\,U(\infty,t_n) \quad \cdots (3)$$

using (3) in (2)

$$\langle 0|\varphi(x_n)\cdots\varphi(x_1)|0\rangle = \langle 0|U^\dagger(\infty,0)U(\infty,t_n)\,\phi_I(x_N)\,U(t_n,t_{n-1})\cdots U(t_1,-\infty)\,U(-\infty,0)|0\rangle$$

Now use part (j)

$$= \langle 0|\varphi\rangle\langle\varphi|\cdots\cdots\phi\rangle\langle\phi|0\rangle$$

$$= \langle\phi\cdots\cdots|\phi\rangle\,|\langle 0|\phi\rangle|^2$$

l) This follows from part k, if we insert T, all's U goes to T

$$\langle 0|T\varphi(x_n)\cdots\phi(x_1)|0\rangle = \langle\phi|T\varphi_I(x_N)\cdots\varphi_I(x_1)\,e^{-i\int d^4x\, \mathcal{H}_I(x)}|\phi\rangle\,|\langle 0|\phi\rangle|^2$$

m) Now, we know that $\langle 0|0\rangle = 1$

$$\langle \phi|\phi\rangle = 1$$

Set all $\underline{\phi(x_i) = 1}$ in $\underline{\text{PART}-l}$, we get

$$\langle 0|0\rangle = \langle \phi|\, T e^{-i\int d^4x\, \mathcal{H}_I(x)}\,|\phi\rangle\, |\langle 0|\phi\rangle|^2$$

$$\Rightarrow |\langle \phi|0\rangle|^2 = \frac{1}{\langle \phi|\, T e^{-i\int d^4x\, \mathcal{H}_I(x)}\,|\phi\rangle}$$

(this is just the factor if we wanted to <u>normalize to 1</u>) (looks like).

✓

Problem # 4 Srednicki - 10.4

Two different methods of solution for this problem.

$$\mathcal{L}_I = \frac{1}{2} g \varphi \partial^\mu \varphi \partial_\mu \varphi$$

Since this interaction act as derivation of ϕ, it makes more sense to go to momentum-space by considering Fourier transform ..

It is trivial to see that going to momentum space means making the substitution:

$$\partial_\mu \varphi \longmapsto i k_\mu \tilde{\varphi} \quad - \quad (*)$$

This result can be proved by taking Fourier transform of ϕ and then taking $\partial_\mu \varphi$ of that.

Recall that in one-dimension F.T, we had something like

F.T of $f'(t)$ was $i\omega F(\omega)$; something similar to this notion.

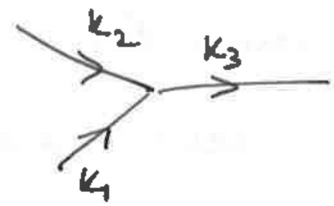
Then, using ① we get

$$\mathcal{L}_I = \frac{1}{2} \; -2! \; g \; k_\mu k^{\mu'}$$

[$i^2 = -1$ is giving the negative sign]

$$= -g \; k_\mu k^{\mu'}$$

where μ & μ' is the permutations of 1, 2, 3



same

$$\mathcal{L}_I = -\frac{g}{2} \left(k_1.k_2 + k_2.k_3 + k_1.k_3 + k_2.k_1 + k_3.k_2 + k_3.k_1 \right)$$

all possible permutations

$$\text{Vertex Factor} = \frac{ig}{2} \left(k_1^2 + k_2^2 + k_3^2 \right)$$

Since $k_1 + k_2 + k_3 = 0$

$(k_1 + k_2 + k_3)^2 = 0$

$k_1^2 + k_2^2 + k_3^2 + 2k_1.k_2 + 2k_2.k_3 + 2k_3.k_1 = 0$

$\Rightarrow k_1^2 + k_2^2 + k_3^2 = -2k_1.k_2 - 2k_2.k_3 - 2k_3.k_1$

SECOND METHOD :

$$\mathcal{L}_1 = \frac{1}{2} g \phi \partial^\mu \varphi \partial_\mu \varphi$$

Taking F.T to 'k' space, we get

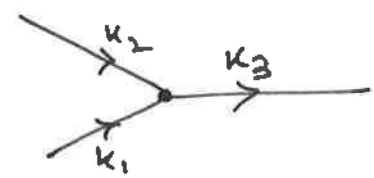
$$\mathcal{L}_I \Big|_{\substack{\text{F.T in} \\ \text{k-space}}} = \frac{1}{2} g \int d^4k_1 \, d^4k_2 \, d^4k_3 \, e^{ik_1x_1} \tilde{\phi}(k_1) \, \partial_2^\mu e^{ik_2x_2} \tilde{\phi}(k_2) \, \partial_3^\mu e^{ik_3x_3} \tilde{\phi}(k_3)$$

giving back our own $\mathcal{L}_I$

Since, we are now in the k space, we write the term contributing to V.F as functional derivatives acting on $\mathcal{L}_I$

$$V.F = i \frac{\delta}{\delta\tilde{\phi}(k_1)} \frac{\delta}{\delta\tilde{\phi}(k_2)} \frac{\delta}{\delta\tilde{\phi}(k_3)} \frac{1}{2} g \int d^4k_1 d^4k_2 d^4k_3 \, e^{ik_1x_1} \tilde{\phi}(k_1) \underline{\partial_2^{\mu} e^{ik_2x_2}} \tilde{\phi}(k_2) \underline{\partial_3^{\mu} e^{ik_3x_3}} \tilde{\phi}(k_3)$$

Taking the underlined derivatives, we get



$$V.F = -i(k_2.k_3) \frac{\delta}{\delta\tilde{\phi}(k_1)} \frac{\delta}{\delta\tilde{\phi}(k_2)} \frac{\delta}{\delta\tilde{\phi}(k_3)} \frac{1}{2} g \int d^4k_1' d^4k_2' d^4k_3' \, e^{ik_1'x_1} \tilde{\phi}(k_1) \, e^{ik_2'x_2} \tilde{\phi}(k_2) \, e^{ik_3'x_3} \tilde{\phi}(k_3)$$ + possible permutations

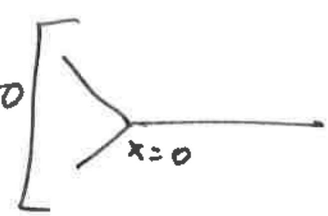
Taking the functional derivatives, they hit $\tilde{\phi}(k_i)$ and pull a δ^4 function

$$= -i(k_2.k_3) \frac{1}{2} g \int d^4k_1' d^4k_2' d^4k_3' \, e^{ik_1'x_1} e^{ik_2'x_2} e^{ik_3'x_3} \delta(k_1-k_1')\delta(k_2-k_2')\delta(k_3-k_3')$$

Now doing the integral over k_1', k_2' & k_3' we get

$$= -i(k_2.k_3) \frac{1}{2} g \, e^{ik_1x_1} e^{ik_2x_2} e^{ik_3x_3}$$

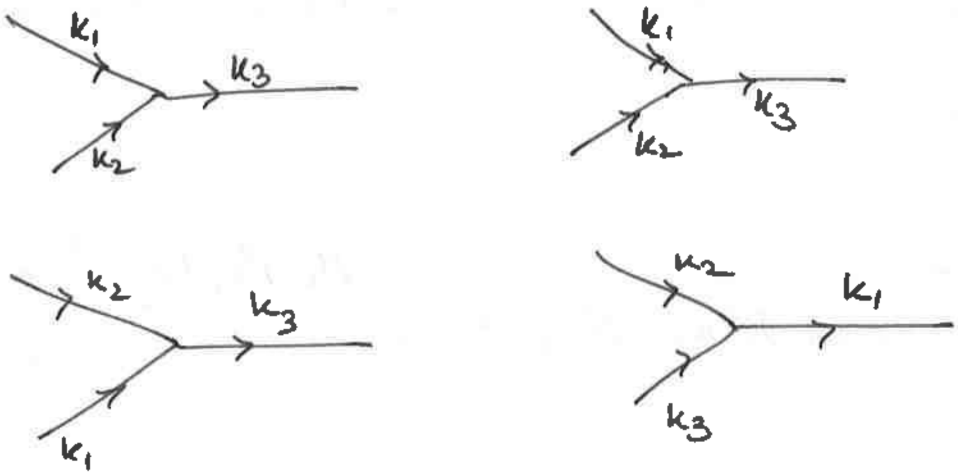
Now let $x_1 = x_2 = x_3 = x$ and place $x = 0$



we get,

$$V.F = -i(k_2.k_3) \frac{1}{2} g$$

But, we have not only one but six diagrams i.e.:



+ <u>2 more</u>..

Adding up all contributions, we get

$$V.F = \frac{-ig}{2}\left(k_2\cdot k_3 + k_3\cdot k_1 + k_1\cdot k_2 + k_2\cdot k_1 + k_1\cdot k_3 + k_3\cdot k_2\right)$$

$$= \frac{+ig}{2}\left(k_1^2 + k_2^2 + k_3^2\right)$$ ✓

Since $k_1 + k_2 + k_3 = 0$

$\Rightarrow k_1^2 + k_2^2 + k_3^2 = -2(k_1\cdot k_2 + k_2\cdot k_3 + k_1\cdot k_3)$

We can also calculate this <u>Vertex Factor</u> in x-space and that is not followed here!!

Problem 4-3 Srednicki 10.5

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{m^2\varphi^2}{2}$$

$$\phi \longmapsto \varphi + \lambda\varphi^2$$

$$\mathcal{L} = -\frac{1}{2}\partial^\mu(\varphi+\lambda\varphi^2)\partial_\mu(\varphi+\lambda\varphi^2) - \frac{1}{2}m^2(\varphi+\lambda\varphi^2)^2$$

$$= \underbrace{-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2}_{\mathcal{L}_0} \underbrace{-\lambda m^2\varphi^3 - \frac{1}{2}\lambda^2 m^2\varphi^4 - 2\lambda\varphi\partial^\mu\varphi\partial_\mu\varphi - 2\lambda^2\varphi^2\partial^\mu\varphi\partial_\mu\varphi}_{\mathcal{L}_1}$$

$$= \mathcal{L}_0 + \mathcal{L}_1$$

Clearly, this theory has both three-point and four-point vertex factors.

$\mathcal{L}_I$		Vertex Factor	What Point vertex
$-2\lambda\phi\partial^\mu\varphi\partial_\mu\varphi$	use Result From 10.4	$-2i\lambda(k_1^2+k_2^2+k_3^2)$	3
$-2\lambda^2\varphi^2\partial^\mu\varphi\partial_\mu\varphi$	Extend idea to φ^4 like	$-4i\lambda^2(k_1^2+k_2^2+k_3^2+k_4^2)$	4
$-\lambda m^2\varphi^3$	From ϕ^3 theory in text	$-6im^2\lambda$	3
$-\frac{1}{2}\lambda m^2\varphi^4$	Exercise Ch. 9.	$\frac{-1}{2}4!\,im^2\lambda^2$	4

Now, adding corresponding 3-point & 4-point together.

* The 3-point vertex gets vertex factor of

$$-2i\lambda(k_1^2+k_2^2+k_3^2) - 6im^2\lambda$$

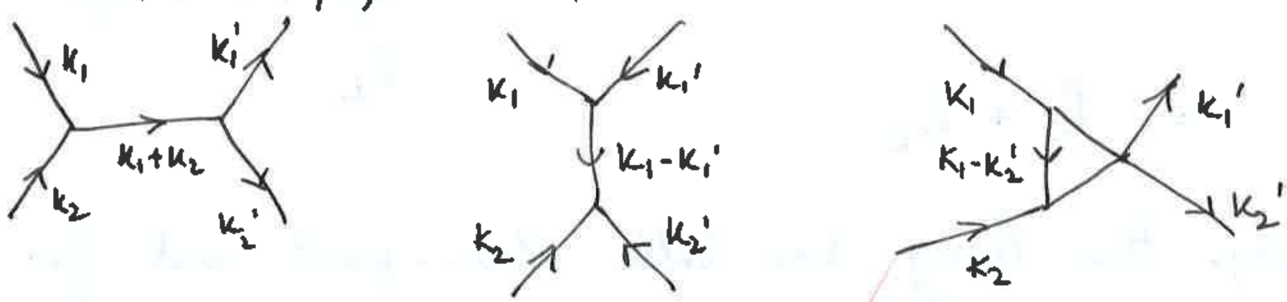
Simplifying,

$$-2i\lambda(k_1^2+k_2^2+k_3^2+3m^2) \quad \text{— (A)}$$

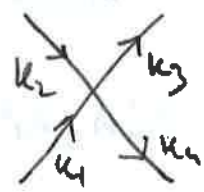
* The 4-point vertex gets vertex factor of

$$-4i\lambda^2(k_1^2+k_2^2+k_3^2+k_4^2) - 12im^2\lambda^2 \quad \text{— (B)}$$

Now $\phi\phi \to \phi\phi$ diagrams are :



s-channel t-channel u-channel



4-point vertex diagram. } 4 external lines

Vertex factor for 4-point vertex was :

$$-4i\lambda^2(k_1^2+k_2^2+k_3^2+k_4^2) - 12im^2\lambda^2$$

$$= -4i\lambda^2\left[k_1^2+k_2^2+k_3^2+k_4^2+3m^2)\right]$$

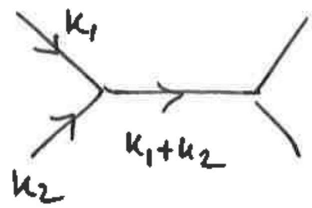
$$= -4i\lambda^2\left[(k_1^2+m^2)+(k_2^2+m^2)+(k_3^2+m^2)+(k_4^2+m^2)-m^2\right]$$

V.F = + 4im^2λ^2

$$V.F = +4im^2\lambda^2$$

Now, using the rules from text; we get

$$i\mathcal{T} = \left[-2i\lambda(-s+m^2)\right]^2 \frac{1}{i(-s+m^2)}\Bigg|_{\text{just for s-channel}}$$



Two-on shell external lines $k_1^2 = m^2$, $k_2^2 = m^2$

let $\vec{k}_1 + \vec{k}_2 = \vec{k}_3$ here $\vec{k}_3^{\,2} = -s$

$$+\left[\quad\right]_{s\to t} \text{ for t channel} + \left[\quad\right]_{s\to u} \text{ for u channel}$$

$$+ 4im^2\lambda^2\Big|_{\text{4-point contribution}}$$

From Ⓐ we get

$-2i\lambda(k_3^2+m^2)$

$-2i\lambda(-s+m^2)$ → one vertex

Internal line

$$\frac{-i}{k_3^2+m^2} = \frac{-i}{-s+m^2} = \frac{1}{i(-s+m^2)}$$

$$i\mathcal{T} = 4i\lambda^2(-s+m^2) + 4i\lambda^2(-t+m^2) + 4i\lambda^2(-u+m^2) + 4im^2\lambda^2$$

$$= 4i\lambda^2(-s-t-u+4m^2)$$

But if all of them have same mass, m (as it is $\phi\phi\to\phi\phi$ here)

Then $\boxed{i\mathcal{T} = 0}$

Problem # 6

$$V(r) \propto \int d^3k \, e^{i\vec{k}\cdot\vec{x}} \frac{1}{\vec{k}^2}$$ { Massless scalar particle exchange).

$$\propto \frac{1}{r}$$ (in 4-D as given in the statement).

(Similar to the Energy term we derived for Yukawa case in class).

Also, Lecture 4.2 Page 5 bottom.

Now, suppose we have 'n+3+1' space-time dimension, where 'n' is arbitrary. 3+1 is the case we did above $V \propto \frac{1}{r}$.

$$V(r)\Big|_{\substack{n+3+1 \\ \text{dimensions}}} \propto \int d^{3+n}k \, e^{i\vec{k}\cdot\vec{x}} \frac{1}{\vec{k}^2}$$

$$\propto \int d^3k \, \underline{d^nk} \, e^{i\vec{k}\cdot\vec{x}} \frac{1}{|\vec{k}|^2}$$

(d^nk) → contributes $\frac{1}{r^n}$

Example:

5 dim → 4 space, 1 time → $V \propto \frac{1}{r^2}$

6 dim → 5 space, 1 time → $V \propto \frac{1}{r^3}$

$$\propto \frac{1}{r} \cdot \frac{1}{r^n}$$

$$\propto \frac{1}{r^{n+1}}$$

Careful, some special cases! $d=3 \sim \log(rm)$

(variation of Pot. energy where n = Extra space dimensions).

60/50

Quantum Field Theory - Assignment 4

Raghav Govind Jha

March 18, 2014

Srednicki 10.2

Looking at Assigment 2, we see from 3.5 that, there are two particles a, b. They are conjugate to each other. If 'a' is a particle with charge 1, then 'b' is antiparticle with -1 charge. Now, ϕ annihilates the 'a' particle, but, it also creates a 'b' particle. Similarly, $\phi^\dagger$ does the opposite. Now, in the last assignment we saw that when we had $J^\dagger$, we made the arrow directed towards it and away if it was J.

It means that when we have an incoming 'a' and outgoing 'b' correspond to external legs where arrows are towards the vertex. And similarly, outgoing 'a' and incoming 'b' will correspond to external legs which has arrow away from vertex. Incoming particles have their momentum that point towards the vertex and vice versa. But, then this is all good for 'a' but not for 'b. We have to reverse the momentum direction for 'b' to encapsulate one arrow for both charge and momentum.

This is somewhat closely related to the rules followed when we draw Feynman diagrams for scattering processes which involves particles and anti-particles (Dirac, QED) !

The Feynman's rules are given below :

- For every incoming 'a' particle, draw a line with an arrow pointed towards the vertex and label the four-momentum.
- For every outgoing 'a' particle, draw a line with an arrow pointed away from the vertex and label the four-momentum.
- For every incoming 'b' particle, draw a line with an arrow pointed away from the vertex and label the four-momentum with a NEGATIVE sign.
- For every outgoing 'b' particle, draw a line with an arrow pointed towards the vertex and label the four-momentum with a NEGATIVE sign.
- The allowed vertex joins four lines where two point away and two point towards it. Draw all diagrams satisfying this (including extra internal lines, if necessary).
- Assign the four-momentum to the internal line by conserving four-momentum at each vertex.
- The diagram now consists of :

- 1. External line (incoming and outgoing particles) – Factor of 1
- 2. For each vertex $-i\lambda$
- 3. For internal line factor of $\frac{-i}{k^2+m^2-i\epsilon}$

- The value of $i\mathcal{T}$ is given by sum over all diagrams.

Srednicki 11.3

a) $\Gamma = \frac{32 G_F^2}{m} \int (k_1 \cdot k_2')(k_1' \cdot k_3') \, dLIPS_3(k_1)$

use eq. 11.23 Srednicki i.e

$$dLIPS_{n'}(k) = (2\pi)^4 \delta^4 \left(k - \sum_{j=1}^{n'} k_j'\right) \prod_{j=1}^{n'} d\tilde{k}_j'$$

i.e $dLIPS_3(k_1) = (2\pi)^4 \delta^4 (k_1 - k_1' - k_2' - k_3') \, d\tilde{k}_1' \, d\tilde{k}_2' \, d\tilde{k}_3'$

$dLIPS_2(k_1 - k_3') = (2\pi)^4 \delta^4 (k_1 - k_1' - k_2' - k_3') \, d\tilde{k}_1' \, d\tilde{k}_2'$

i.e we can rewrite Γ as

$$\Gamma = \frac{32 G_F^2}{m} \int (k_1 \cdot k_2')(k_1' \cdot k_3') \, dLIPS_2(k_1 - k_3') \, d\tilde{k}_3'$$

$$= \frac{32 G_F^2}{m} \int d\tilde{k}_3' \, k_{1\mu} k_{3\nu}' \int k_2'^{\mu} k_1'^{\nu} \, dLIPS_2(k_1 - k_3')$$

✓

b) $\int k_1'^{\mu} k_2'^{\nu} \, dLIPS_2(k) = A k^2 g^{\mu\nu} + B k^{\mu} k^{\nu}$

L.H.S has two indices μ & ν. Also the R.H.S has to be Lorentz invariant. The R.H.S cannot depend on k_1^{μ} or k_2^{ν}. only possible two indices can be constructed are from $g^{\mu\nu}$ (invariant) & $k^{\mu} k^{\nu}$ (to maintain rank 2 contravariant).

$g^{\mu\nu} \rightarrow$ Dimensionless. L.H.S has dimensions of k^2 i.e $(mass)^2$. We take a linear combination of $g^{\mu\nu}$ and $k^\mu k^\nu$ i.e

$$\int k_1'^{\mu} k_2'^{\nu} \, dLIPS_2(k) = A k^2 g^{\mu\nu} + B k^\mu k^\nu$$

where A, B are dimensionless numbers.

c) Show that for $m_1' = m_2' = 0$,

$$dLIPS_2(\bar{k}) = 1/8\pi$$

Use Srednicki eq. 11.30 i.e

$$dLIPS_2(k) = \frac{|\vec{k}_1'|}{16\pi^2\sqrt{s}} \, d\Omega_{CM}$$

$$\sqrt{s} = \sqrt{(k_1+k_2)^2}$$; here $k_2 = 0$, $k_1 = m_\mu$

$= m_\mu$.

Also, using 11.3 we see that

$$|\vec{k}'| = \frac{1}{2\sqrt{s}} \sqrt{s^2 - 2(m_1'^2 + m_2'^2)s + (m_1'^2 - m_2'^2)}$$

$$\Rightarrow \sqrt{s}/2 \Rightarrow m_\mu/2$$

$$dLIPS_2(k) = \frac{\cancel{m_\mu}}{2}\frac{1}{16\pi^2}\frac{1}{\cancel{m_\mu}}\,d\Omega_{CM}$$

$$= \frac{1}{32\pi^2}\,d\Omega_{CM} \Rightarrow \frac{1}{8\pi}$$ ✓

d) $$\int k_1'^{\mu} k_2'^{\nu}\, dLIPS_2(k) = Ak^2 g^{\mu\nu} + Bk^{\mu}k^{\nu}$$

Contracting with $g_{\mu\nu}$, we get

$$\int (k_1' \cdot k_2')\, dLIPS_2(k) = 4Ak^2 + Bk^2$$ $\underline{g^{\mu\nu}g_{\mu\nu} = 4}$

$$k_1'\cdot k_2' = \frac{k^2 - k_1'^2 - k_2'^2}{2}$$

$$\Rightarrow k_1'\cdot k_2' = \frac{k^2}{2} \quad \left[k_1'^2 = k_2'^2 = 0 \quad \text{where } A = \cancel{\sqrt{\tfrac{m_\mu}{2}}}\right]$$

$$\frac{1}{2}k^2\int dLIPS_2(k) = 4Ak^2 + Bk^2$$

$$\frac{1}{2}\cancel{k^2}\,\frac{1}{16\pi} = (4A + B)\cancel{k^2} \qquad - ①$$

Contracting with $k_\mu k_\nu$ we get,

$$\int (k_1'\cdot k)(k_2'\cdot k)\, dLIPS_2(k) = Ak^4 + Bk^4$$

$$\int \frac{1}{2}k^2\frac{1}{2}k^2 \int dLIPS_2(k) = (A + B)k^4$$

$$\Rightarrow A + B = \frac{1}{32\pi} \qquad - ②$$

From ① & ②, we get

$$A = \frac{1}{96\pi}$$

$$B = \frac{1}{48\pi}$$ ✓

e) From part Ⓐ we have

$$\Gamma = \frac{32 G_F^2}{m} \int d\tilde{k}_3 \; k_{1\mu} k'_{3\nu} \int k_2'^{\mu} k_1'^{\nu} \, dLIPS_2(k_1 - k_3') \quad \cdots ①$$

But $\int k_1'^{\mu} k_2'^{\nu} \, dLIPS_2(k) = \frac{1}{96\pi} k^2 g^{\mu\nu} + \frac{1}{48\pi} k^{\mu} k^{\nu}$ $\cdots ②$

use ② in ①

$$\Gamma = \frac{32 G_F^2}{m} \int d\tilde{k}_3' \; k_{1\mu} k'_{3\nu} \left(\frac{1}{96\pi} (k_1 - k_3')^2 g^{\mu\nu} + \frac{1}{48\pi} (k_1 - k_3')^{\mu} (k_1 - k_3')^{\nu} \right.$$

$$= \frac{G_F^2}{3\pi m} \int d\tilde{k}_3' \; k_{1\mu} k'_{3\nu} \left((k_1 - k_3')^2 g^{\mu\nu} + 2 (k_1 - k_3')^{\mu} (k_1 - k_3')^{\nu} \right)$$

$$= \frac{G_F^2}{3\pi m} \int d\tilde{k}_3' \; k_{1\mu} k'_{3\nu} \Big((k_1^2 + k_3'^2 - 2 k_1 \cdot k_3') g^{\mu\nu} + 2 k_1^{\mu} k_1^{\nu} - 2 k_1^{\mu} k_3'^{\nu} - 2 k_3'^{\mu} k_1^{\nu} + 2 k_3'^{\mu} k_3'^{\nu}$$

But we have been given that

$k_1 = (m, \tilde{0})$ i.e $k_1^2 = -m^2$

and $k_3'^2 = 0$

And also given that $k_3' = E_e = E_3'$

i.e $k_1 \cdot k_3' = -m E_e$

$$\Gamma = \frac{G_F^2}{3\pi m} \int d\tilde{k}_3' \, k_{1\mu} k_{3\nu}' \Big((-m^2 + 0 + 2m E_e) g^{\mu\nu} + 2 k_1^\mu k_1^\nu - 2 k_1^\mu k_3'^\nu - 2 k_3'^\mu k_1^\nu + 2 k_3'^\mu k_3'^\nu \Big)$$

$$= \frac{G_F^2}{3\pi m} \int d\tilde{k}_3' \Big((-m^2 + 2m E_e) k_{1\mu} k_{3\nu}' g^{\mu\nu} + 2 k_{1\mu} k_{3\nu}' k_1^\mu k_1^\nu - 2 k_{1\mu} k_{3\nu}' k_1^\mu k_3'^\nu - 2 k_{1\mu} k_{3\nu}' k_3'^\mu k_1^\nu + 2 k_{1\mu} k_{3\nu}' k_\mu' k_3'^\nu \Big)$$

$$= \frac{G_F^2}{3\pi m} \int d\tilde{k}_3' \Big((-m^2 + 2m E_e)(k_1 \cdot k_3') + 2k_1^2 (k_1 \cdot k_3') - 2(k_1^2)(k_3'^2) - 2(k_1 \cdot k_3')^2 + 2(k_1 \cdot k_3')(k_3')^2 \Big)$$

Again use the result that

$$\left. \begin{aligned} k_1 \cdot k_3' &= - m E_e \\ k_3^2 &= 0 \; ; \\ k_1^2 &= - m^2 \end{aligned} \right\}$$

$$= \frac{G_F^2}{3\pi m}\int d\tilde{k}_3' \left((+m^3 E_e - 2m^2 E_e^2) + 2m^3 E_e - 2m^2 E_e^2 \right.$$

$$= \frac{G_F^2}{3\pi m}\int d\tilde{k}_3' \left(3m^3 E_e - 4m^2 E_e^2 \right)$$

$$= \frac{m G_F^2}{\pi}\int d\tilde{k}_3' \left(E_e m - \frac{4}{3} E_e^2 \right)$$

But $d\tilde{k}_3' = \dfrac{d^3 k_3'}{(2\pi)^3\, 2E_e}$ ⟹ Lorentz Invariant differential

$$= \frac{m G_F^2}{\pi}\int \frac{d^3 k_3'}{(2\pi)^3\, 2E_e} \left(E_e m - \frac{4}{3} E_e^2 \right)$$

$k_3' \Leftrightarrow E_e$

Volume element $d^3 k_3' \Leftrightarrow d^3 E_e$

$d^3 k_3' \Leftrightarrow d^3 E_e$

$$= \frac{m G_F^2}{\pi (2\pi)^3} (2\pi) \int \left(E_e^2 m - \frac{4}{3} E_e^3 \right) dE_e$$

$$\frac{d\Gamma}{dE_e} = \frac{m G_F^2}{4\pi^3} \left(E_e^2 m - \frac{4}{3} E_e^3 \right)$$

Set $\frac{d\Gamma}{dE_e} = 0$

we get,

$$\frac{mG_F^2}{4\pi^3}\left(2E_e m - 4E_e^2\right) = 0$$

↓ ZERO

$\Rightarrow$ $\boxed{E_e = \frac{m}{2}}$ $\boxed{(E_e)_{max} = \frac{m}{2}}$ ✓

f) $$\frac{d\Gamma}{dE_e} = \frac{mG_F^2}{4\pi^3}\left(E_e^2 m - \frac{4}{3}E_e^3\right)$$

$$\Gamma = \frac{mG_F^2}{4\pi^3}\int_0^{m/2} dE_e\left(E_e^2 m - \frac{4}{3}E_e^3\right)$$ [Since this is max. value of E_e)

$$= \frac{mG_F^2}{4\pi^3}\left[\frac{E_e^3 m}{3} - \frac{4E_e^4}{3\times 4}\right]_0^{m/2}$$

$$= \frac{mG_F^2}{4\pi^3}\left[\frac{m^3 m}{8\times 3} - \frac{1}{3}\frac{m^4}{16}\right]$$

$$= \frac{mG_F^2}{4\pi^3}\left[\frac{m^4}{24} - \frac{m^4}{48}\right]$$

$$= \frac{m^5 G_F^2}{192\pi^3}$$ ✓

g) $\tau = \frac{1}{\Gamma} \Rightarrow \Gamma = \frac{1}{2.197 \times 10^{-6} s}$

$= 4.55 \times 10^{5} s^{-1}$

Now in Planck units

$\Gamma = 4.55 \times 10^{5} s^{-1}$

$1 s^{-1} = \frac{1}{1.5192 \times 10^{24}} \text{GeV}$

$\Gamma = \frac{4.55 \times 10^{5}}{1.5192 \times 10^{24}}$

$\simeq 3 \times 10^{-19} \text{ GeV}$

From part (f)

$G_F = \sqrt{\frac{192 \pi^3 \Gamma}{m^5}}$

$= \sqrt{\frac{192 \pi^3 \times (3 \times 10^{-19}) \text{GeV}}{(.105 \text{ GeV})^5}}$

$= 1.164 \times 10^{-5} \text{ GeV}^{-2}$ ✓

↳ almost the Fermi constant

h) $P(E_e) = \frac{1}{\Gamma}\frac{d\Gamma}{dE_e}$

$$= \frac{192\pi^3}{m^5 G_F^2}\cdot\frac{m G_F^2}{4\pi^3}\left(E_e^2 m - \frac{4}{3}E_e^3\right)$$

$$= \frac{48}{m}\left(\frac{E_e^2}{m^2} - \frac{4}{3}\frac{E_e^3}{m^3}\right)$$

Let $\frac{E_e}{m} = x$

$$P(E_e/m) = 48\left(x^2 - \frac{4}{3}x^3\right)$$

$$P(x) = 48\left(x^2 - \frac{4}{3}x^3\right)$$

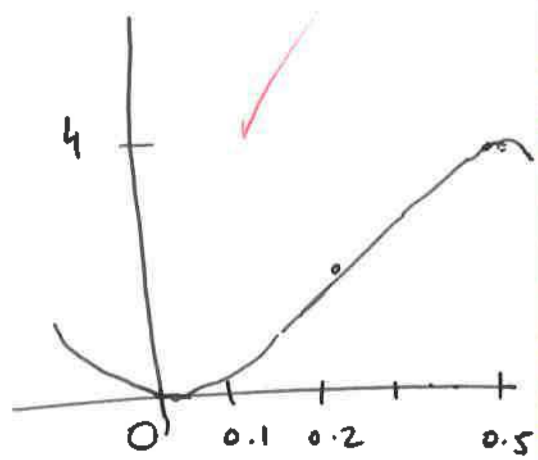
$P(0) = 0$

$P(1) = 48\left(1 - \frac{4}{3}\right) = -16$

$P\left(\frac{1}{2}\right) = 48\left(\frac{1}{4} - \frac{4}{3}\frac{1}{8}\right)$

$= 48\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{48 \times 2}{24}$

$= 4$



Srednicki 11.1

using 11.49 of the textbook,

$$\Gamma = \frac{1}{S}\int d\Gamma$$

But $d\Gamma$ is given by eq. 11.48

$$= \frac{1}{S}\int \frac{1}{2E_1} |\mathcal{T}|^2 \, dLIPS_2(k_1)$$

use eq. 11.30 for $dLIPS_2(k)$

$$= \frac{1}{S}\int \frac{1}{2E_1} |\mathcal{T}|^2 \frac{|\vec{k}_1'|}{16\pi^2\sqrt{s}} \, d\Omega_{CM}$$

$|\vec{k}_1'|$ is given by eq. 11.2

$$= \frac{1}{S}\int \frac{1}{2E_1} |\mathcal{T}|^2 \frac{\frac{1}{2\sqrt{m_1^2}}\sqrt{m_1^4 - 2(m_1'^2+m_2'^2)m_1^2 + (m_1'^2-m_2'^2)^2}}{16\pi^2\sqrt{s}} \, d\Omega \quad \left[s = m_1^2 = E_1^2 \right.$$

$$= \frac{1}{S}\frac{1}{64\pi^2 m_1^3}\int |\mathcal{T}|^2 \sqrt{m_1^4 - 2(m_1'^2+m_2'^2)m_1^2 + (m_1'^2-m_2'^2)^2} \, d\Omega$$

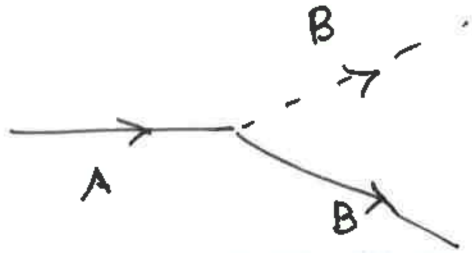
In this case, we have $A \to B + B$

i.e $m_1' = m_2' = m_B$

$$\Gamma = \frac{1}{S}\frac{1}{64\pi^2 m_A^3}\int |\mathcal{T}|^2 \sqrt{m_A^4 - 4m_A^2 m_B^2} \, d\Omega$$

$$= \frac{1}{S} \frac{1}{64\pi^2 m_A} \sqrt{1 - \frac{4 m_B^2}{m_A^2}} \int |\mathcal{M}|^2 \, d\Omega$$

Now, we need to find matrix element and S



$\longrightarrow$ Vertex factor $= (S)\, ig$

here $S = 2$ (flipping B's)

$$\mathcal{L}_1 = g A B^2 \quad \checkmark$$

Therefore, we have

$$\Gamma = \frac{\cancel{4} g^2}{32\,\cancel{128}\,\pi^2 m_A} \sqrt{1 - \frac{4 m_B^2}{m_A^2}} \int d\Omega$$

$$= \frac{1}{32\pi^2 m_A} g^2 \int \sqrt{1 - \frac{4 m_B^2}{m_A^2}} \, d\Omega$$

But $\int d\Omega = 4\pi$, we get

$$\Gamma = \frac{g^2}{8\pi m_A} \sqrt{1 - \frac{4 m_B^2}{m_A^2}}$$

b) $\mathcal{L}_1 = g \phi \chi^{+} \chi$

this clearly amounts to changing

$A \to \phi$

$B \to \chi$ and $B \to \chi^{+}$ before.

From Last part,

$$\Gamma = \frac{1}{S} \frac{1}{64\pi^2 m_A} \sqrt{1 - \frac{4m_B^2}{m_A^2}} \int |\mathcal{T}|^2 d\Omega$$

Now $S = 1$ (only) can't flip χ and χ^+

and $|\mathcal{T}|^2 = g^2$

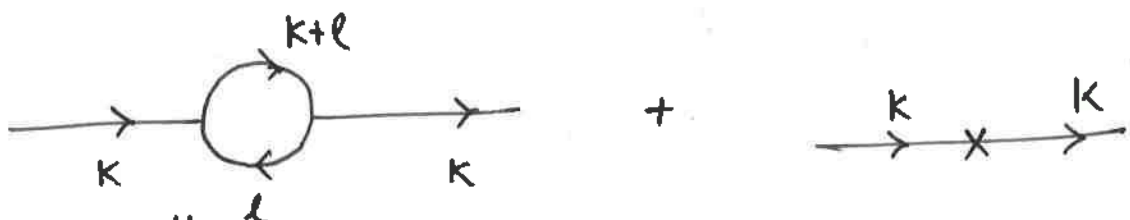
we have

$$\Gamma = \frac{1}{64\pi^2 m_\phi} g^2 \sqrt{1 - \frac{4 m_\chi^2}{m_\phi^2}} \int d\Omega$$

$$= \frac{g^2}{16\pi^2 m_\phi} \sqrt{1 - \frac{4m_\chi^2}{m_\phi^2}}$$ ✓ Ans.

Srednicki 14.5 $O(\lambda)$ correction to propagator in φ^4 theory.

In the text, Srednicki does the $O(g^2)$ correction to the propagator in φ^3 theory.



⇓

$P = 4, \ V = 2$

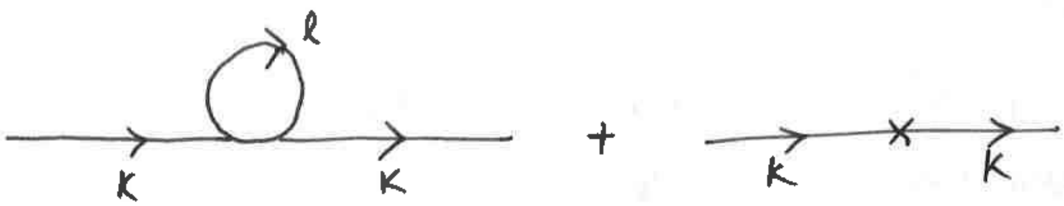
$E = 2P - 3V$
$= 8 - 6 = 2$ ✓ Remember.

Now, this diagram changes for ϕ^4 theory

$$E = 2P - 4V$$

$$2 = 2P - 4V$$

we have, $P = 3, V = 1$ i.e the corrections to propagator in ϕ^4 theory.



We note the Feynman's rule for the self-energy diagram and add contribution of second diagram (counterterm). Feynman's rule will give us $i\Sigma$ or $i\Pi$. So, we multiply by $-i$ to get Π

$$\Pi(k^2) = (-i)\left(\frac{-i\lambda}{2}\right)\int \frac{d^d l}{(2\pi)^d} \frac{-i}{l^2+m^2-i\epsilon} + (-i)(-i)(Ak^2 + Bm^2)$$

$$i\Pi(k^2) = -\frac{\lambda}{2}\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2+m^2-i\epsilon} - Ak^2 - Bm^2$$

Let's Wick's rotate this

$$l^0 = i\bar{l}_d$$

$$d^d l = i d^d \bar{l}$$

$$i\Pi(k^2) = \frac{-i\lambda}{2}\int \frac{d^d\bar{l}}{(2\pi)^d}\frac{1}{\bar{l}^2+m^2} - Ak^2 - Bm^2$$

$\boxed{\epsilon \to 0}$ for evaluation

Use Srednicki 14.27, also this theory is renormalizable only when $d = 4$. since only then λ is dimensionless.

From 14.27, we get

$$\Pi(k^2) = \frac{-\lambda}{2}\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}(m^2)^{-1+d/2} - Ak^2 - Bm^2 \quad \ldots (3)$$

We use the same trick as Srednicki used.

$$\epsilon \equiv 4-d$$

$$\lambda \to \tilde{\mu}^\epsilon \lambda$$ (all mass dimension projected to $\tilde{\mu}$

$$\Pi(k^2) = \frac{-\lambda\tilde{\mu}^\epsilon}{2}\frac{\Gamma(-1+\epsilon/2)}{(4\pi)^2}(4\pi)^{\epsilon/2}(m^2)^{1-\epsilon/2} - Ak^2 - Bm^2$$

$$= \frac{-m^2\lambda}{2(4\pi)^2}\Gamma\left(-1+\frac{\epsilon}{2}\right)\left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)^{\epsilon/2} - Ak^2 - Bm^2 \quad \ldots (4)$$

Now use Srednicki 14.26 and 14.33

$$\Gamma\left(-1+\frac{\epsilon}{2}\right) = \frac{(-1)^{+1}}{1!}\left[\frac{1}{\epsilon/2} - \gamma + 1\right] \quad \cdots\cdots \quad (5)$$

using (5) in (4) we get

$$\Pi(k^2) = \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + 1\right]\left[1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

Keep $O(\epsilon)$

$$= \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + 1 + \ln\left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

Now, take clue from <u>14.35 Srednicki</u> i.e

$$\mu \equiv \sqrt{4\pi}\, e^{-\gamma/2}\, \tilde{\mu}$$

<u>OR</u>

$$4\pi\tilde{\mu}^2 = e^{\gamma}\mu^2 \quad ✓.$$

$$= \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + 1 + \ln\left(\frac{e^{\gamma}\mu^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

$$= \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon} - \cancel{\gamma} + 1 + \cancel{\ln(e^{\gamma})} + \ln\left(\frac{\mu^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

$$= \frac{m^2\lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right)\right] - Ak^2 - Bm^2$$

Now this has to be finite and independent of $\underline{\mu}$.

expand A in λ i.e

$$A = \xi \lambda + O(\lambda^2)$$

Also $\underline{\Pi'(-m^2) = 0}$ Required

where $'$ is derivative w.r.t k^2, this gives us that $-\xi \lambda = 0 \Rightarrow \boxed{\xi = 0}$, so $\underline{A = O(\lambda^2)}$ ✓

$$\Pi(k^2) = \frac{m^2 \lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right)\right] - Bm^2$$

Now, we choose B to encapsulate the effect in it

$$B = \frac{\lambda}{2(4\pi)^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right)\right] + \lambda k_B + O(\lambda^2)$$

We get $\Pi(k^2) = -\lambda k_B m^2 + O(\lambda^2)$

Imposing $\boxed{\Pi(-m^2) = 0} \Rightarrow \underline{k_B = 0}$

Then,

$$B = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\epsilon} + \frac{1}{2} + \ln\left(\frac{\mu}{m}\right)\right] + O(\lambda^2)$$

↗ some 4π, γ_E missing here ... other finite terms

$$\Pi(k^2) = O(\lambda^2)$$

And $\Pi(k^2) = 0$ to $O(\lambda)$

Surprising.. No correction !!

you adjust counterterms ↑ so not so surprising

Problem #4 $\mathcal{L} = -\frac{1}{2}\left((\partial_\mu \varphi)^2 + m^2 \varphi^2\right) + Z_\lambda \frac{\lambda}{4!} \varphi^4 + \mathcal{L}_{ct}$

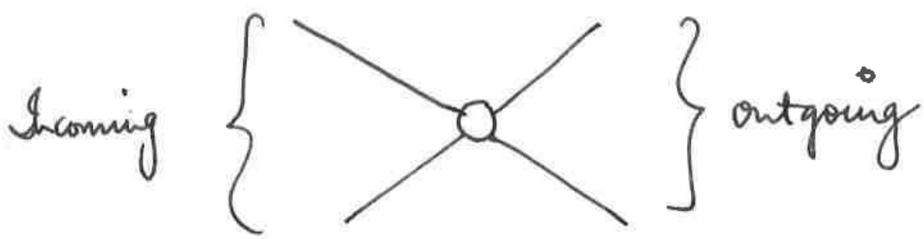
At tree level $\mathcal{L}_{ct} \to 0$

$Z_\lambda = 1$, we get

$$\mathcal{L} = -\frac{1}{2}\left((\partial_\mu \varphi)^2 + m^2 \varphi^2\right) + \frac{\lambda \varphi^4}{4!}$$

$2 \longrightarrow 2$ Scattering at tree level (no loops)

The only contributing diagram here is



Tree diagram

We can write down the Feynman's rule and find $|\mathcal{T}|^2$. We have four external lines (all factor of 1) and one vertex.

So $\mathcal{T} = -i\lambda$

$$\boxed{|\mathcal{T}|^2 = \lambda^2}$$ ✓

In φ^3 theory, we have s-channel, u-channel and t-channel at tree level. But here its just one single diagram.

Now, following the same derivation as in text we see that for $2 \to 2$ scattering, we have

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2 s} \frac{|\vec{k}_1'|}{|\vec{k}_1|} |\mathcal{T}|^2$$

if we let $\left.\begin{array}{l} m_1 = m_1' \\ m_2 = m_2' \end{array}\right\}$ we get $|\vec{k}_1'| = |\vec{k}_1|$

$2 \to 2$ scattering (of same masses)

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2 s} |\mathcal{T}|^2$$

where $s = (E_1 + E_2)^2$ in CM frame

and $|\mathcal{T}|^2 = \lambda^2$ [as found from diagram]

So, we get

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2 s} \lambda^2$$

$$d\sigma = \frac{\lambda^2}{64\pi^2 s} d\Omega_{CM}$$

Total cross section σ ;

$$\sigma = \frac{\lambda^2}{64\pi^2 s} \int d\Omega_{CM}$$

$$= \frac{\lambda^2}{16\pi^2 s}$$

Problem 6: Srednicki 14.7

$$L = \frac{1}{2} Z \dot{q}^2 - \frac{1}{2} Z_\omega \omega^2 q^2 - Z_\lambda \lambda \omega^3 q^4$$

Set $\hbar = 1 = m$, $\lambda \rightarrow$ dimensionless.

a) Conjugate momentum = $\frac{\partial L}{\partial \dot{q}} = Z\dot{q} = P$

$$H = P\dot{q} - L$$

$$= \frac{1}{2} Z \dot{q}^2 + \frac{1}{2} Z_\omega \omega^2 q^2 + Z_\lambda \lambda \omega^3 q^4$$

$$= \frac{1}{2} Z^{-1} P^2 + \frac{1}{2} Z_\omega \omega^2 Q^2 + Z_\lambda \lambda \omega^3 Q^4$$

$$H = \underbrace{\frac{1}{2} P^2 + \frac{\omega^2 Q^2}{2}}_{H_0} + \underbrace{\frac{1}{2}(Z^{-1}-1)P^2 + \frac{1}{2}(Z_\omega - 1)\omega^2 Q^2 + Z_\lambda \lambda \omega^3 Q^4}_{H_I}$$

Energy of the state (unperturbed) given by

$$E_i = \left(i + \frac{1}{2}\right)\omega$$

First-order correction given by:

$$E_i^1 = \langle i | H_1 | i \rangle \Longrightarrow \text{QM Perturbation Theory Result.}$$

$$E_i^1 = \langle i | \tfrac{1}{2}(Z^{-1}-1)P^2 + \tfrac{1}{2}(Z_\omega - 1)\omega^2 Q^2 + Z_\lambda \lambda \omega^3 Q^4 | i \rangle$$

Write $Z^{+1} = 1 + A$; $Z_\omega = 1 + B$, we get
$Z^{-1} \cong 1 - A$

$$E_i^1 = \langle i | \tfrac{-1}{2} A P^2 + \tfrac{1}{2} B \omega^2 Q^2 + \lambda \omega^3 Q^4 | i \rangle$$

Now use $A = k_A \lambda + O(\lambda^2)$
$B = k_B \lambda + O(\lambda^2)$

$$E_i^1 = \tfrac{-1}{2} k_A \lambda \langle i | P^2 | i \rangle + \tfrac{1}{2} k_B \lambda \omega^2 \langle i | Q^2 | i \rangle + \lambda \omega^3 \langle i | Q^4 | i \rangle + O(\lambda^2)$$

Now recall, creation and annihilation operators of QM.

$$\hat{P} = i \sqrt{\frac{\omega}{2}} \left(a^{+} - a\right)$$

$$\hat{Q} = \frac{1}{\sqrt{2\omega}} \left(a^{+} + a\right)$$

$$E_i^1 = \frac{\omega\lambda}{4} k_A \langle i | (a^{+} - a)^2 | i \rangle + \frac{\omega\lambda}{4} k_B \langle i | (a^{+}+a)^2 | i \rangle + \frac{\lambda \omega^3}{4\omega^2} \langle i | (a^{+}+a)^4 | i \rangle$$

$\longrightarrow$ we used the normalization given by Srednicki

$$E_i^1 = \frac{-\omega\lambda}{4} k_A \langle i | aa^+ + a^+a | i \rangle$$

$\left(\text{Since} \langle i | (a^+)^2 | i \rangle = 0 \right.$ Orthogonality

$\left. \langle i | (a)^2 | i \rangle = 0 \right)$

$$+ \frac{\omega\lambda}{4} k_B \langle i | aa^+ + a^+a | i \rangle$$

$$+ \frac{\lambda\omega}{4} \langle i | aaa^+a^+ + aa^+aa^+ + aa^+a^+a + a^+aaa^+ + a^+a^+aa^+ + a^+a^+aa | i \rangle$$

${}^4C_2$ combination i.e $\frac{4!}{2!\,2!} = 6$ ✓

When $i = 0$ (ground state)

$$E_0^1 = \frac{-\omega\lambda}{4} k_A + \frac{\omega\lambda}{4} k_B + \frac{3\lambda\omega}{4} + O(\lambda^2)$$

Hence, the ground state energy of perturbed system is:

$$E_\Omega = \frac{\omega}{2} + \frac{\lambda\omega}{4}(-k_A + k_B + 3) + O(\lambda^2)$$

When $i = 1$ (excited state)

$\langle 1 | aa^+ + a^+a | 1 \rangle$

$= \langle 1 | aa^+ + 1 | 1 \rangle$

$= \langle 1 | 1 \rangle 3$

$= 3$ ✓

$$E_1^1 = \frac{-3\omega\lambda}{4} k_A + \frac{3\omega\lambda}{4} k_B + \frac{\lambda\omega}{4}(6+4+2+2+1+0)$$

$$= \frac{-3\omega\lambda}{4} k_A + \frac{3\omega\lambda}{4} k_B + \frac{15\lambda\omega}{4}$$

$$E_I = \frac{3\omega}{2} + \frac{3\omega\lambda}{4}\left(-k_A + k_B + 5\right)$$

Now, we need to correct the eigenstates. They are given by:

$$|n\rangle = |n^0\rangle + \sum_{n \neq m} \frac{\langle m | H' | 0 \rangle}{E_0^0 - E_m^0} |m\rangle$$

→ Reference: Griffiths QM

$$|\Omega\rangle = |0\rangle + \sum_{n \neq m} \frac{1}{\frac{\omega}{2} - (m+\frac{1}{2})\omega} \Big[-\frac{1}{2} k_a \lambda \langle m | P^2 | 0 \rangle + \frac{1}{2} k_B \lambda \omega^2 \langle m | Q^2 | 0 \rangle + \lambda \omega^3 \langle m | Q^4 | 0 \rangle \Big] |m\rangle$$

Now use P^2 and Q^2 and Q^4

$$= |0\rangle + \sum_{n \neq m} \frac{\lambda}{4m} \Big[-k_a \langle m | (a^\dagger - a)^2 | 0 \rangle - k_B \langle m | (a^\dagger + a)^2 | 0 \rangle - \langle m | (a^\dagger + a)^4 | 0 \rangle \Big] |m\rangle$$

In $(a^\dagger - a)^2$ and $(a^\dagger + a)^2$ only $a^\dagger a^\dagger$ and $a^\dagger a$ survives since $n \neq m$

$$= |0\rangle + \sum_{n \neq m} \frac{\lambda}{4m} \Big[-k_a \langle m | a^\dagger a^\dagger | 0 \rangle - k_B \langle m | a^\dagger a | 0 \rangle - \langle m | (a^\dagger + a)^4 | 0 \rangle \Big] |m\rangle$$

First two term are *only* non-zero if $m=2$

$$|\Omega\rangle = |0\rangle + \frac{\lambda}{8}\left[-k_A\langle 2|a^\dagger a|0\rangle - k_B\langle 2|a^\dagger a|0\rangle\right]|m\rangle$$

$$- \sum_{n\neq m}\frac{\lambda}{4m}\langle m|(a^\dagger+a)^4|0\rangle|m\rangle$$

$$= |0\rangle - \frac{\lambda(k_A+k_B)}{8}\sqrt{2}\,|2\rangle - \sum_{n\neq m}\frac{\lambda}{4m}\langle m|(a^\dagger+a)^4|0\rangle|m\rangle$$

only for $m=2$ or $m=4$

$$= |0\rangle - \frac{\sqrt{2}\lambda(k_A+k_B)}{8}|2\rangle - A|2\rangle - B|4\rangle$$

Where $A = \frac{\lambda}{4m}\langle m|(a^\dagger+a)^4|0\rangle$ for $m=2$

$B = \frac{\lambda}{4m}\langle m|(a^\dagger+a)^4|0\rangle$ for $m=4$

~~$A = \frac{\lambda}{4m\times 2}\langle 2|(a^\dagger)^4 + (a)^4 +$~~

we use

$$\langle n'|(a^\dagger+a)^4|n\rangle = \sqrt{(n+4)(n+3)(n+2)(n+1)}\,\delta_{n',n+4}$$
$$+(4n+6)\sqrt{(n+2)(n+1)}\,\delta_{n',n+2}$$
$$+(6n^2+6n+3)\,\delta_{n',n}$$
$$+(4n-2)\sqrt{n(n-1)}\,\delta_{n,n+2}$$
$$+\sqrt{n(n-1)(n-2)(n-3)}\,\delta_{n',n+4}$$

if $n' = 4$, $n = 0$, we get as A

$$A = \frac{6\sqrt{2}\,\lambda}{8} \quad ; \quad B = \frac{\sqrt{24}\,\lambda}{8}$$

we get,

$$|\Omega\rangle = |0\rangle - \frac{\sqrt{2}\lambda(K_A + K_B + 6)}{8}|2\rangle - \frac{\sqrt{6}\lambda}{8}|4\rangle + O(\lambda^2)$$

Same process for $|I\rangle$ gives $|1\rangle$, $|3\rangle$ and $|5\rangle$ states

$$|I\rangle = |1\rangle - A'|3\rangle - B'|5\rangle + O(\lambda^2)$$

c) $$\omega = E_I - E_\Omega$$

$$= \frac{3\omega}{2} + \frac{3\omega\lambda}{4}(-K_A + K_B + 5) - \frac{\omega}{2} - \frac{\lambda\omega}{4}(-K_A + K_B + 3)$$

$$= \omega + \frac{\omega\lambda}{4}(-3K_A + 3K_B + 15 + K_A - K_B - 3)$$

$$= \omega + \frac{\omega\lambda}{2}(-K_A + K_B + 6) = \omega$$

$$\Rightarrow \quad K_B - K_A = -6$$

$$\Rightarrow \boxed{K_A - K_B = 6}$$

Similarly using $\langle I | Q | \Omega \rangle = \frac{1}{\sqrt{2\omega}}$ gives second condition, and we can find k_A & k_B.

$k_A = 0$; $k_B = -6$

d) $L = \frac{1}{2} Z \dot{q}^2 - \frac{1}{2} Z_\omega \omega^2 q^2 - Z_\lambda \lambda \omega^3 q^4$

Write again by seperating $\frac{1}{2}\dot{q}^2$ and $\frac{1}{2}\omega^2 q^2$

$$L = \underbrace{\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2}_{\text{free}} + \underbrace{\frac{1}{2}(Z-1)\dot{q}^2 - \frac{1}{2}(Z_\omega - 1)\omega^2 q^2}_{\text{counter}} \underbrace{- Z_\lambda \lambda \omega^3 q^4}_{\text{interaction}}$$

This will admit similar loop corrections as ϕ^4 theory we did before

here Z_λ is not with $\frac{1}{4!}$, So we will have

[S=2]

$$i\Pi_1(k^2) = \frac{1}{2}\int \frac{dl}{2\pi} \frac{24\lambda\omega^3}{l^2+\omega^2-i\epsilon}$$

Where $24\lambda\omega^3$ is from interaction q^4 terms.

Doing again Wick rotation, we get

$$\Pi_1(k^2) :\equiv \frac{24\lambda\omega^3}{4\pi}\int \frac{1}{\ell^2+\omega^2}\,d\ell \qquad \left[d^d\ell = d\ell \text{ since } d=1\right]$$

$$= \frac{6\lambda\omega^3}{\pi}\int \frac{1}{\ell^2+\omega^2}\,d\ell$$

→ Contour integration gives $\frac{\pi}{\omega}$

↓ standard integral

$$= \frac{6\lambda\omega^3}{\pi}\,\frac{\pi}{\omega}$$

$$= 6\lambda\omega^2 \quad \text{(This is the contribution of 1st diagram)} \quad \ldots\ (A)$$

<u>Contribution of second term</u> i.e

$$\Pi_2(k^2) = -\left((Z-1)k^2 - (Z_\omega-1)\omega^2\right) \quad \left\{ \begin{array}{l} S=2 \\ \text{cancels } \tfrac{1}{2} \\ \text{in front} \end{array} \right.$$

Let $Z-1 = K_a\lambda$

$Z_\omega - 1 = K_B\lambda$

$$\Pi_2(k^2) = -\lambda\left(K_a k^2 - K_B\omega^2\right) \quad \text{— (B)}$$

Combining (A) & (B), we have total

$$\Pi(k^2) = 6\lambda\omega^2 - \lambda\left(K_a k^2 - K_B\omega^2\right)$$

Now we calculate $\Pi'(k^2)$ and $\Pi(-m^2)$

$$\Pi'(k^2) = -\lambda K_a = 0 \quad \Rightarrow \quad \boxed{K_a = 0}$$

$$\Pi(-m^2) = 6\lambda\omega^2 + \lambda K_B \omega^2 = 0$$

$$\Rightarrow \boxed{K_B = -6}$$

So, we see that $K_B = -6$ matches from Part C,

Quantum Field Theory - Assignment 5

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April 8, 2014

57/60

Problem 1 Srednicki 16.1

Solution : Attached hand written at the end.

Problem 2 Srednicki 22.1

Solution : The Noether current is defined as :

$$j^\mu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \varphi_a(x))}\delta\varphi_a(x)$$

And then the Noether charge, Q is defined as :

$$Q \equiv \int d^3x \; j^0(x)$$

$$Q \equiv \int d^3x \; \frac{\partial \mathcal{L}(x)}{\partial(\partial_0 \varphi_a(x))}\delta\varphi_a(x)$$

$$Q \equiv \int d^3x \; \frac{\partial \mathcal{L}(x)}{\partial\dot{\varphi}_a(x)}\delta\varphi_a(x)$$

$$Q = \int d^3x \; \Pi^a(x)\delta\varphi_a(x) \tag{1}$$

We have to evaluate $[\varphi_a, Q]$

$$[\varphi_a, Q] = [\varphi_a, \int d^3x \; \Pi^a(x)\delta\varphi_a(x)] \tag{2}$$

$$= \int d^3x \;\; [\varphi_a, \Pi^b(x)]\delta\varphi_b(x)\delta_{ab} \tag{3}$$

Using the commutation relation,

$$[\Pi(\mathbf{x},t), \varphi(\mathbf{x}',t)] = -i\delta^3(\mathbf{x}-\mathbf{x}')$$

Evaluating (3) and doing the δ integration, we get

$$[\varphi_a, Q] = i\delta\varphi_a$$

Problem 3 : Consider the Lagrangian for N complex scalar fields ϕ^a, with a $= 1, 2, ...N$:

$$\mathcal{L} = \partial_\mu \phi_a^\dagger \partial^\mu \phi^a - V(|\phi|)$$

with $|\phi^2| = \sum_a \phi_a^\dagger \phi^a$

a) Find the global symmetries of $\mathcal{L}$?

b) Use Noether's theorem to compute the conserved currents associated with this symmetry ?

c) Construct the charges. In quantum theory, find their commutators ?

O(2N) → larger symmetry group

Solution : This given Lagrangian is invariant under global U(N) symmetry. It is because the lagrangian consists of N scalar fields and their adjoint and such a unitary group transformation leaves the lagrangian invariant. The symmetry is global since the parameter α which we define below does not depend on space-time coordinates.

The scalar fields transforms as

O(2N) has $\frac{2N(2N-1)}{2}$ generators

$$\phi_a = \mathcal{U}_a{}^b \phi_b$$

$\boxed{2N^2 - N}$ $\neq N^2$

where $\mathcal{U}$ is given by :

$$\mathcal{U} = e^{i\alpha^m \mathcal{T}_m}$$

$2N^2 - N - N^2 \Rightarrow N^2 - N \Rightarrow N(N-1)$ → This should be no. of Goldstone bosons.

$$\phi'(x) = e^{i\alpha^m \mathcal{T}_m} \phi(x) \tag{4}$$

where, $\mathcal{T}$ are N $\times$ N hermitian matrices (generators of the U(N) group).

Note that any $N\times$ N unitary matrix can be written in terms of hermitian matrix $\mathcal{T}$ as :

$$U = \exp\left(i\alpha^k \mathcal{T}_k\right)$$

To convince ourselves, we can check that :

$$\exp\left(i\alpha^k \mathcal{T}_k\right)^\dagger \exp\left(i\alpha^k \mathcal{T}_k\right) = 1 = UU^\dagger$$

We used the fact that $\mathcal{T}$ was a hermitian matrix.

This symmetry group has N^2 generators. This means that we will have N^2 conserved currents and charges. It is important to note that the symmetry group related to EM interactions is U(1) which has just a single generator. Also, U(N) can be thought as $U(N) = U(1) \bigotimes SU(N)$. The U(1) group has one generator (photon !) and SU(N) has $N^2 - 1$ generators and they add up neatly to N^2.

$$\phi'(x) \equiv \phi(x) + i\alpha\mathcal{T}\phi(x)$$

This gives us,

$$\delta\phi'(x) = i\alpha\mathcal{T}\phi(x)$$

Now, the Noether current is given by :

} Check the last pages.

$$j^\mu = \frac{\partial L}{\partial(\partial_\mu \phi)} \delta\phi'(x)$$

$$j^\mu = i\alpha \frac{\partial L}{\partial(\partial_\mu \phi)} T\phi(x)$$

To press the fact that we have same number of currents as generator, we specify it with an index.

$$j_i^\mu = i\alpha \frac{\partial L}{\partial(\partial_\mu \phi)} T_i\phi(x)$$

The Noether charge is given by :

$$Q_i \equiv \int d^3x\, j_i^0$$ Redone ! Later.

This gives us,

$$Q_i \equiv i\alpha \int d^3x\, \Pi\, T_i\phi(x)$$

Now, we also recall our old commutation relation,

$$[\Pi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = -i\delta^3(\mathbf{x} - \mathbf{x}')$$

Corrected Part(b) and Part(c) on the ~~next~~ last few pages

Problem 4 Srednicki 36.5

Solution : N massless Weyl fields ψ_j

$$\mathcal{L} = i\psi_j^\dagger \sigma^\mu \partial_\mu \psi_j$$

where the repeated index j is summed. The lagrangian is clearly invariant under the U(N) transformation,

$$\psi_j \to U_{jk}\psi_k$$

where U is a unitary matrix. State the invariance group of the following cases :

a) N Weyl fields with a common mass m,

$$\mathcal{L} = i\psi_j^\dagger \sigma^\mu \partial_\mu \psi_j - \frac{1}{2}m\left(\psi_j\psi_j + \psi_j^\dagger\psi_j^\dagger\right)$$

★

Let's make the transformation :

$$\psi_j \to A_{ji}\psi_i$$

where, A will be identified belonging to the group such that the lagrangian is invariant under the above change of ψ.

$$\mathcal{L} \to i(A_{ji}\psi_i)^\dagger \sigma^\mu \partial_\mu (A_{ji}\psi_i) - \frac{1}{2}m(A_{ji}\psi_i)(A_{ji}\psi_i) - \frac{1}{2}m(A_{ji}\psi_i)^\dagger (A_{ji}\psi_i)^\dagger$$

$$\mathcal{L} \to i(\psi_i^\dagger A_{ji}^\dagger \sigma^\mu \partial_\mu A_{ji}\psi_i) - \frac{1}{2}m(\psi_i A_{ij}^T A_{ji}\psi_i) - \frac{1}{2}m(\psi_i^\dagger A_{ij}^* \psi_i^\dagger A_{ji}^\dagger)$$

Now, is clear to see that the invariant transformation must be real matrices since we have a complex conjugate in the second part which cannot be cancelled if we don't consider it to be real.

This changed lagrangian will give the initial if following is true :

- $A = A^*$
- $A^\dagger A = \mathcal{I}$
- $AA^T = \mathcal{I} = A^T A$

These are the properties of real unitary matrix which is all called orthogonal. **Hence this is invariant under O(N)** ✓

b) N massless Majorana fields,

$$\mathcal{L} = \frac{i}{2}\Psi_j^T \mathcal{C}\gamma^\mu \partial_\mu \Psi_j$$

★

We can re-write the given Lagrangian in the following form :

$$\mathcal{L} = \frac{i}{2}\bar{\Psi}_j\gamma^\mu\partial_\mu\Psi_j$$

We have used the result that $\Psi^T\mathcal{C} = \bar{\Psi}_j$

Now, using Srednicki (36.27) and neglecting the **boundary term**, we obtain :

$$\mathcal{L} = \frac{i}{2}\left[\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \zeta^\dagger\bar{\sigma}^\mu\partial_\mu\zeta\right]$$

Now, we recall that for Majorana fields, particles are their own anti-particles and we obtain :

$$\mathcal{L} = i\left(\chi_j^\dagger\sigma^\mu\partial_\mu\chi\right)$$

Comparing the above with the lagrangian for N-massless Weyl fields, we see that they are the same. Hence, they transform similarly. **They transform under U(N)**.

c) N Majorana fields with a common mass m,

$$\mathcal{L} = \frac{i}{2}\Psi_j^T\mathcal{C}\gamma^\mu\partial_\mu\Psi_j - \frac{1}{2}m\Psi_j^T\mathcal{C}\Psi_j$$

★

This case is similar to the situation in part(a) where, the we can do similar stuff with what we did in the previous part. This lagrangian will be invariant under same group as in part(a). This is **invariant under O(N)**.

d) N massless Dirac fields

$$\mathcal{L} = i\bar{\Psi}_j\gamma^\mu\partial_\mu\Psi_j$$

★

Each Dirac field consists of two Weyl spinors. This lagrangian is that of 2N massless Weyl fields and it remains invariant under U(2N). Also, note that in part(b), the lagrangian was invariant under U(N) since they were Majorana massless fields where particle and antiparticle were same. But, here there is an increase of 2-fold symmetry because of the mixing and we have a lagrangian that is invariant under **U(2N)** (enlarged group).

e) N Dirac fields with common mass m,

$$\mathcal{L} = i\bar{\Psi}_j\gamma^\mu\partial_\mu\Psi_j - m\bar{\Psi}_j\Psi_j$$

★

This is now almost identical to Part(c) done above for N massless Majorana fields. Here, we will have the **symmetry group as O(2N)** since, there are twice as many terms, infact 2N-massive Weyl spinors.

Problem 5 Srednicki 33.2

The Lorentz group (homogenous) consists of boost and rotation. If **K** and **J** are the boost and the angular momentum operators, then we can write :
The Lie algebra for this $SO(3,1)$ homogenous Lorentz group is :

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k$$

We define N_i and $N_i^\dagger$ as below :

$$N_i = \frac{1}{2}(J_i - iK_i)$$

$$N_i^\dagger = \frac{1}{2}(J_i + iK_i)$$

Now, the commutation relations of N_i and $N_i^\dagger$ are,

$$[N_i, N_j] = i\epsilon_{ijk}N_k \quad (5)$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger \quad (6)$$

$$[N_i, N_j^\dagger] = 0 \quad (7)$$

This implies that

- **N** and **N**† are independent and have intrinsic SU(2) symmetry.

And Lorentz group has SU(2) $\bigotimes$ SU(2) symmetry (same algebra) following this with 6 independent parameters (i.e 3 for boosts and 3 for angular momentum).

We have to prove Eq. 1-3 using the above defined relations.

Proof 1 :

$$[N_i, N_j] = \frac{1}{4}[J_i - iK_i, J_j - iK_j]$$

$$4[N_i, N_j] = [J_i, J_j] + [J_i, -iK_j] - i[K_i, J_j] - [K_i, K_j]$$

This gives us using above defined relations

$$4[N_i, N_j] = i\epsilon_{ijk}\left(J_k - iK_k - iK_k + J_k\right)$$

$$4[N_i, N_j] = 4i\epsilon_{ijk}N_k$$

$$\Longrightarrow$$

$$[N_i, N_j] = i\epsilon_{ijk}N_k$$

Proof 2 :

$$[N_i^\dagger, N_j^\dagger] = \frac{1}{4}[J_i + iK_i, J_j + iK_j]$$

$$4[N_i^\dagger, N_j^\dagger] = [J_i, J_j] + [J_i, iK_j] + i[K_i, J_j] - [K_i, K_j]$$

This gives us using above defined relations

$$4[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}\left(J_k + iK_k + iK_k + J_k\right)$$

$$4[N_i, N_j] = 4i\epsilon_{ijk}N_k^\dagger$$

$$\Longrightarrow$$

$$[N_i, N_j] = i\epsilon_{ijk}N_k^\dagger$$

Proof 3 :

$$[N_i, N_j^\dagger] = \frac{1}{4}[J_i - iK_i, J_j + iK_j]$$

$$4[N_i, N_j] = [J_i, J_j] + [J_i, iK_j] - i[K_i, J_j] + [K_i, K_j]$$

This gives us using above defined relations

$$4[N_i, N_j] = i\epsilon_{ijk}\left(J_k + iK_k - iK_k - J_k\right)$$

$$4[N_i, N_j] = 0$$

$$\Longrightarrow$$

$$[N_i, N_j] = 0$$

Problem 6 Srednicki 34.2

Solution : We need to show that Eq (34.9) and Eq (34.10) obey Eq (34.4)

$$(S_L^{ij})_a{}^b = \frac{1}{2}\epsilon_{ijk}\sigma_k \quad (34.9)$$

$$(S_L^{k0})_a{}^b = \frac{1}{2}i\sigma_k \quad (34.10)$$

$$[S_L^{\mu\nu}, S_L^{\rho\sigma}] = i\left(g^{\mu\rho}S_L^{\nu\sigma} - g^{\nu\rho}S_L^{\mu\sigma} - g^{\mu\sigma}S_L^{\nu\rho} + g^{\nu\sigma}S_L^{\mu\rho}\right) \quad (34.4)$$

Writing in terms of same indices as we need to prove, 34.4 reads as :

$$[S_L^{ij}, S_L^{mn}] = i\left(g^{im}S_L^{jn} - g^{jm}S_L^{in} - g^{in}S_L^{jm} + g^{jn}S_L^{im}\right)$$

Now, using (34.9) and taking the commutator with i, j & m, n indices, we have :

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\epsilon^{ijk}\epsilon^{mnp}[\sigma_k, \sigma_p]$$

Now if we have $k = p$ above then, the commutator will vanish. But, we can play around with taking $k = m$ and $k = n$. This gives us,

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(\epsilon^{ijm}\epsilon^{mnp}[\sigma_m, \sigma_p] + \epsilon^{ijn}\epsilon^{mnp}[\sigma_n, \sigma_p]\right)$$

Now doing the same thing with using $p = i$ and $p = j$ we obtain eventually,

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(\epsilon^{ijm}\epsilon^{mni}[\sigma_m, \sigma_i] + \epsilon^{ijm}\epsilon^{mnj}[\sigma_m, \sigma_j] + \epsilon^{ijn}\epsilon^{mni}[\sigma_n, \sigma_j] + \epsilon^{ijn}\epsilon^{mnj}[\sigma_n, \sigma_j]\right) \quad (8)$$

Now, this looks pretty much set for matching what we expect.
Recall that,

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma^c = 4iS_L^{ab}$$

Using this in Eq(8) and noting that the delta-function that comes in can be written as metric tensor. We obtain, the following :

$$[S_L^{ij}, S_L^{mn}] = i\left(g^{im}S_L^{jn} - g^{jm}S_L^{in} - g^{in}S_L^{jm} + g^{jn}S_L^{im}\right)$$

This concludes Part(a). ✓

Part B: Show that

$$(S_L^{k0})_a{}^b = \frac{1}{2}i\sigma_k \quad (34.10)$$

obeys 34.4 as well.
We take the commutator constructed of (34.10) as,

$$[S^{k0}, S^{l0}] = -\frac{1}{4}[\sigma_k, \sigma_l]$$

Using the commutator of $\sigma' s$ we get,

$$[S^{k0}, S^{l0}] = \frac{-i}{2}\epsilon_{klm}\sigma_m = -iS_L^{kl} \quad (9)$$

Now, we go back to (34.4) and see what we should expect from that in this case.

$$[S^{k0}, S^{l0}] = i\left(g^{kl}S_l^{00} - g^{0l}S_L^{k0} - g^{k0}S_L^{0l} + g^{00}S_L^{kl}\right)$$

Note the following things : 1) S is antisymmetric tensor and 2) Metric is diagonal. The first term is zero, second and third are zero too ! And, these reduce the above to :

$$[S^{k0}, S^{l0}] = -iS_L^{kl} \quad (10)$$ ✓

Comparing (9) and (10), we prove the required.

k0 with lm? (boosts w/ rotations?)

3.6 $\mathcal{L} = \partial_\mu \varphi_a^+ \partial^\mu \varphi^a - V(|\varphi|)$

we proved that it is invariant under $U(N)$

In this case

$$\phi_i' = e^{i(\alpha.T)_{ik}} \phi_k$$

$$\simeq \phi_k + i\,(\alpha.T)_{ik}\,\phi_k$$

$$\Rightarrow \delta\phi_i = \phi_i' - \phi_i$$

$$= i\,(\alpha.T)_{ik}\,\phi_k \quad — \quad ①$$

and $\delta\phi_i^+ = -i\,(\alpha.T)_{ik}\,\phi_k^+ \quad — \quad ②$

Now we write the expression for δL

$$\delta L = \frac{\partial \mathcal{L}}{\partial \varphi_i}\delta\varphi_i + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\delta(\partial_\mu\varphi_i)} + \frac{\partial \mathcal{L}}{\partial \varphi_i^+}\delta(\varphi_i^+) + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i^+)}\delta(\partial_\mu\varphi_i^+)}$$

Breaking the underlined terms, we get

$$= \frac{\partial \mathcal{L}}{\partial \varphi_i}\delta\phi_i + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\delta\phi_i\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\right)\delta\phi_i$$

$$+ \frac{\partial \mathcal{L}}{\partial \varphi_j^+}\delta\phi_j^+ + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j^+)}\delta\phi_j^+\right)$$

$$- \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_j^+)}\right)\delta\phi_j^+$$

underlined term $\to 0$, since they are EOM

$$\delta L = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\delta\phi_i\right) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i^+)}\delta\phi_i^+\right)$$

we can write this as

$$\delta L = \alpha_a \partial^\mu J_\mu^a \quad \text{where } a = 1 \dots N^2$$

if $\delta L = 0$; then $\partial^\mu J_\mu^a = 0$

J_μ^a → conserved current for symmetry of Lagrangian.

we have

$$J_a^\mu = i\left(\partial^\mu \varphi^+ T_a \varphi - \varphi^+ T_a \partial^\mu \varphi\right) \quad — \ (4)$$

we will have $\underline{N^2}$ conserved currents and as many charges. ✓

c) Now $Q = \int d^3x \, J^0$

But here we have

$$Q_a = \int d^3x \, J_a^0 \quad \text{where } J_a^0 \text{ is given by } (4)$$

$$\boxed{Q_a = \int d^3x \; i\left(\pi^+ T_a \phi - \phi^+ T_a \pi\right)}$$

$$\boxed{Q_b = \int d^3x \; i\left(\pi^+ T_b \phi - \phi^+ T_b \pi\right)}$$

Calculate their commutator

$$[Q_a, Q_b] = -\int d^3x\, [\pi^\dagger T^a \phi, \pi^\dagger T^b \phi] + \int d^3x\, [\pi^\dagger T^a \phi, \phi^\dagger T^b \pi]$$

$$+ \int d^3x\, [\phi^\dagger T^a \pi, \pi^\dagger T^b \phi^b]$$

$$- \int d^3x\, [\phi^\dagger T^a \pi, \phi^\dagger T^b \pi]$$

we now treat ϕ and $\phi^\dagger$ as independent fields i.e $[\phi, \phi^\dagger] = 0$ $\quad [\pi(x), \phi(x)] = -i\delta^3(x-x')$

$$[\phi, \phi] = 0$$

Doing the above commutator, we get

$$[Q_a, Q_b] = [T_a, T_b]$$

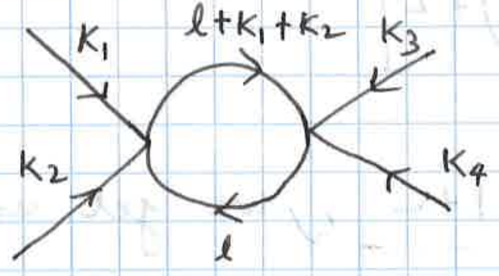
$$= i f_{abc} T^c \times \quad i f_{abc} Q_c$$

not T ↙ operators on both sides

$$= i f_{abc} Q_c$$ → structure constant of $U(N)$

Problem 1 ϕ^4 theory - Loop corrections

Compute $O(\lambda^2)$ correction to V_4. Take $V_4 = -\lambda$. $\underline{s = 4m^2}$



+ 2 other diagrams ✓

$O(\lambda^2)$ correction to V_4. Two other diagrams are obtained by doing $k_2 \leftrightarrow k_3$ and $k_2 \leftrightarrow k_4$.

These correspond to $s - u$ & t - channels.

Srednicki talks about this in Chapter 31

We use Srednicki 31.8 directly.

$$V_4 = -Z_\lambda \lambda + \frac{1}{2}\lambda^2\left[F(-s) + F(-t) + F(-u)\right] + O(\lambda^3) \quad - (1)$$

where $s = -(k_1+k_2)^2$

$t = -(k_1+k_3)^2$

$u = -(k_1+k_4)^2$

and $$F(k^2) = \frac{1}{16\pi^2}\left[\frac{2}{\epsilon} + \int_0^1 \ln(\mu^2/D)\,dx\right] \quad - (2)$$

and $D = x(1-x)k^2 + m^2$

Now, we are given $s = 4m^2$, means $\underline{t = u = 0}$ ✓

We see that

$$F(0) = \frac{1}{16\pi^2}\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right)\right]$$

$$S = -k^2 = 4m^2$$

and $$F(-4m^2) = \frac{1}{16\pi^2}\left[\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + 2\right]$$

Now, lets put $Z_\lambda = 1+C$, we get using ①

$$V_4 = -(1+C)\lambda + \frac{\lambda^2}{2} F(-4m^2) + O(\lambda^3)$$

But $V_4 = -\lambda$ (given)

$$-\lambda = -\lambda - C\lambda + \frac{\lambda^2}{2} F(-4m^2)$$

$$C \Rightarrow \frac{\lambda}{2} F(-4m^2)$$

But when we include the contribution of two other diagrams, we get

$$C = \frac{3\lambda}{16\pi^2}\left[\frac{1}{\epsilon} + \ln\left(\frac{\mu}{m}\right) + \frac{1}{3}\right] + O(\lambda^2)$$

1. $L = \frac{\dot{x}^2}{2} - \frac{w^2 x^2}{2} - g x^3 - h x^4$

$m = 1$, $\hbar = 1$

Sol^n: We know that S (action) and $\hbar$ (reduced Planck's constant) have same units.

$$[S] = [\hbar] = 1 \quad [\text{here}]$$

$$\hbar = \text{Energy} \otimes \text{Time} \quad [\text{Dimension}]$$

$$\Rightarrow \quad [E] = \frac{1}{[t]} \quad - \quad (1)$$

Now $$L \sim \frac{\dot{x}^2}{2} \sim \frac{[x]^2}{[t]^2}$$

$$[L\,t] \sim [S] \sim 1 \sim \frac{[x]^2}{[t]} \cong 1$$

$$\Rightarrow \quad x = [t]^{1/2}$$

From (1) we get

$$\boxed{x = [E]^{-1/2}}$$

$$\boxed{L \sim \frac{[x]^2}{[t]^2} \sim \frac{[E]^{-1}}{[E]^{-2}} \sim [E]}$$

$$\omega^2 x^2 = [E]$$

$$\omega^2 [E]^{-1} = [E]$$

$$\boxed{\omega = [E]}$$

$$[g x^3] = [E]$$

$$\boxed{[g] = \frac{[E]}{[x]^3} = \frac{[E]}{[E]^{-3/2}} = [E]^{5/2}}$$

$$\boxed{h = \frac{[E]}{[x]^4} = \frac{[E]}{[E]^{-2}} = [E]^3}$$

1.2) Clearly, h & g are not dimensionless.

$$[h] = [E]^3 \quad ; \quad [g] = [E]^{5/2}$$

Dimensionless conditions yield;

$$h_{new} = \frac{h_{old}}{\omega^3} \ll 1$$

$$g_{new} = \frac{g_{old}}{\omega^{5/2}} \ll 1$$

So, in this condition, our perturbation theory will be valid.

1.3) $L = \frac{\dot{\varphi}^2}{2} - \frac{\omega^2 \varphi^2}{2} - g\varphi^3 - h\varphi^4$

$x \longmapsto \varphi$

The interaction term has both cube and quartic dependence on φ.

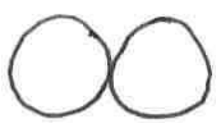
we'll look for diagrams which have no external legs i.e $E=0$ and construct the required three diagrams.

φ^3 theory $\quad E = 2P - 3V$

if $E=0$; nothing possible with $V=1$.

φ^4 theory $\quad E = 2P - 4V$

if $V=1$, $P=2$ then $E=0$



we get other two diagrams due to φ^3 dependence

$P=3$
$V=2$
$E=0$

$P=3$
$V=2$
$E=0$

They are not UV or IR divergent because we have a mass term, coefficient of φ^2

Mass presence $\implies$ IR safe

Also, $D = 0+1 = 1$ i.e we have $\frac{dk}{2\pi}$ measure

and loops contribute as $\sim \frac{1}{k^2}$

i.e Power of 'k' in the numerator < Power of 'k' in denominator

$\implies$ UV Safe.

The scenario will change if we consider $D=4$.

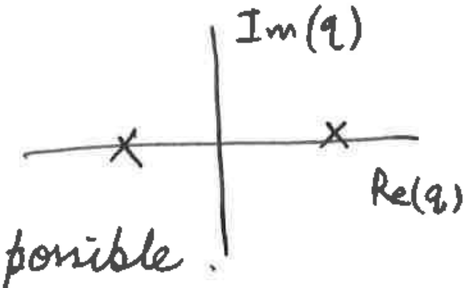
This is valid here since $D=1$ [UV & IR safe !!]

✓

1.4) In the limit $\omega \to 0$. we are spoiling the mass term in $\mathcal{L}$. It will result in no mass (massless) $\implies$ IR divergent.

Also, intuitively $\omega \to 0$ means $k \to 0$ i.e $\lambda \to \infty$, a look at EM spectrum tells that it should be IR divergent (hence the name)..

1.5) In the $\lim_{\omega \to 0}$, the poles are hitting the real 'q' plane and Wick rotating seems not-possible.

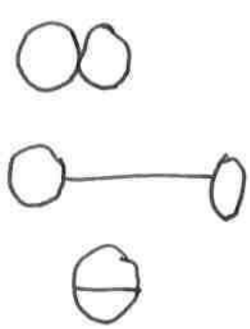


Remedy: Add $i\epsilon$ and save the poles to be on real 'q' axis

$$\frac{1}{k^2 - \omega^2} \xrightarrow{\lim_{\omega \to 0}} \frac{1}{k^2 - i\epsilon}$$

We therefore regulate the integral by adding small 'ϵ' which can be later put to zero.

1.6) Let's go back to the diagrams we had for the problem



Symmetry factor 'S'

$\frac{1}{8}$

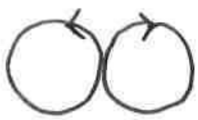
$\frac{1}{8}$

$\frac{1}{12}$

} Recall previous assignments.

Propagator $\qquad \dfrac{i}{k^2-\omega^2}$

Loop integrals $\qquad \dfrac{dk}{2\pi}$



There's no internal line here connecting the two loop so, we'll calculate due to one of them and square it.

Symmetry factor (→ $\frac{1}{8}$)

Since we have h not $\frac{h}{4!}$ (→ $4!$)

$$iM = \frac{1}{8} \times 4! \; i h \int \frac{dk}{(2\pi)} \frac{i}{k^2-\omega^2+i\epsilon}$$

NOTE: The factor of $4!$ here is just because couplings are without factor of $n!$ like QFT we multiply to take this into account. In the next two diagrams, we'll have to multiply by $3!$ similarly !!

$$iM = \frac{1}{8}\, 4!\, \frac{(i)^2 \hbar}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{k^2 - \omega^2 + i\epsilon}$$

Let's Wick rotate this now i.e

$$k = i\tilde{k}$$

$$k^2 = -\tilde{k}^2$$

$$dk \longrightarrow i\, d\tilde{k}$$

$$iM = \frac{1}{8}\, 4!\, \frac{(i^3)\hbar}{(2\pi)^2} \left[\int \frac{d\tilde{k}}{-\tilde{k}^2 - \omega^2}\right]^2 \qquad [\epsilon \to 0]$$

$$= \frac{1}{8}\, 4!\, \frac{-i\hbar}{(2\pi)^2} \left[\int \frac{d\tilde{k}}{-\tilde{k}^2 - \omega^2}\right]^2$$

$$= \frac{4 \times 3 \times 2}{8\; 4} \frac{-i\hbar}{4\pi^2} \left(\frac{\pi}{\omega}\right)^2 \qquad \text{[We used the given integral]}$$

$$= -i \frac{3}{4} \frac{h}{\omega^2}$$

$$M = \frac{-3}{4} \frac{h}{\omega^2}$$

But

$$\Delta E = -M = \frac{3}{4} \frac{h}{\omega^2}$$

→ Call later ✓

First chunck of ΔE

Now, we move to the second diagram

$$S = \frac{1}{8}$$

Now, we have $P = 3$; $V = 2$

$$iM = \frac{1}{8} \times (3!)^2 \, i^2 g^2 \int \frac{dk_1}{(2\pi)} \frac{dk_2}{(2\pi)} \frac{i}{k_1^2-\omega^2} \frac{i}{k_2^2-\omega^2} \frac{i}{-\omega^2}$$

For the internal line, we have $k = 0$, since conservation of momentum means k_1 going to left loop and k_2 going to right one.

$$= +\frac{1}{8} \times \frac{\cancel{6}^{\,3} \times \cancel{6}^{\,3}}{\cancel{4}\pi^2\omega^2} g^2 i \left(\frac{\pi}{\omega}\right)^2$$

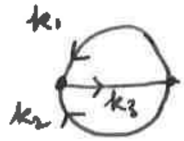
Integrating over k_1 and k_2 & using the given result

$$= +i \frac{9}{8} \frac{g^2}{\cancel{\pi^2}} \frac{\cancel{\pi^2}}{\omega^4}$$

$$\boxed{iM = i\frac{9g^2}{8\omega^4}} \rightarrow \boxed{M = +\frac{9}{8}\frac{g^2}{\omega^4}}$$

use later

Now, the second diagram for φ^3 dependence i.e 3^{rd} diagram in overall evaluating is  with $S = \frac{1}{12}$

For  we have $V = 2$; $P = 3$

Note that the middle propagator depends on two entering propagator with $\vec{k}_1$ and $\vec{k}_2$. We'll pull off a δ-f^n with conservation of momentum at vertices with factor of $(2\pi)^d \delta(k_1 + k_2 - k_3)$; but since $\underline{d = 1}$; we get $\underline{(2\pi)\,\delta(k_1 + k_2 - k_3)}$ ..

$$iM = \frac{1}{12} \times \overset{3!}{6} \times \overset{3!}{6}\,(ig)^2 \frac{i^3}{(2\pi)^3}\overset{2\pi}{\cdot}\int dk_1\,dk_2 \frac{1}{k_1^2 - \omega^2}\,\frac{1}{k_2^2 - \omega^2}\,\frac{1}{(k_1 + k_2)^2 - \omega^2}$$

we have already put $\vec{k}_3 = \vec{k}_2 + \vec{k}_1$ in ↗

$$= \frac{6 \times 6}{12}(ig)^2 \frac{i^3}{(2\pi)^2}\int dk_1\,dk_2 \frac{1}{k_1^2 - \omega^2}\,\frac{1}{k_2^2 - \omega^2}\,\frac{1}{(k_1 + k_2)^2 - \omega^2}$$

Again do the Wick rotation and get the value of integral from the $\underline{\text{given listed}}$ integral, we get

$$= \frac{\overset{3}{\cancel{6} \times \cancel{6}}}{\cancel{2} \times 12}\,\frac{i^2 g^2 i^3}{(2\pi)^2}\left(\frac{\pi^2}{3\omega^4}\right)$$

$$= \frac{3}{12}\, i\, \frac{g^2}{\pi^2}\,\frac{\pi^2}{\omega^4} = i\,\frac{3}{12}\,\frac{g^2}{\omega^4}$$

Now this is

$$iM = i\,\frac{3}{2}\,\frac{g^2}{\omega^4}$$

$$\Rightarrow \boxed{M = \frac{3}{12}\,\frac{g^2}{\omega^4}}$$

Now we add $\underline{M}$'s from  & 

$$M = \frac{3g^2}{12\omega^4} + \frac{9}{8}\,\frac{g^2}{\omega^4}$$

$$= \frac{6g^2 + 27g^2}{24\omega^4}$$

$$= \frac{{}^{11}\cancel{33}\,g^2}{{}_{8}\cancel{24}\,\omega^4}$$

$$= \frac{11}{8}\,\frac{g^2}{\omega^4}$$

But $\Delta E = -M$

$$= \frac{-11}{8}\,\frac{g^2}{\omega^4}$$

[Contribution from ~~quartic~~ cubic dependence interaction]

$$\Delta E_{total} = \frac{-11}{8}\,\frac{g^2}{\omega^4} + \frac{3}{4}\,\frac{h}{\omega^2}$$

PROBLEM 2 : SCALARS

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$$

$$\mathcal{L}_{free} = \frac{1}{2}\left[\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + \partial_\mu C \partial^\mu C - m_A^2 A^2 - m_B^2 B^2 - m_C^2 C^2\right]$$

$$-\mathcal{L}_{int} = \frac{1}{2}\left[g_A A C^2 + g_B B C^2\right]$$

1) Clearly $[\mathcal{L}] = [M]^4$ we are in $d=4$ here..

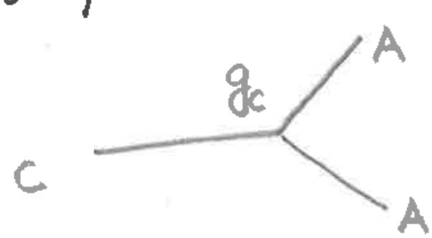
$[A] = [B] = [C] = [M]^1$ the scalar fields have mass dimension 1.

$$\mathcal{L} = [M]^4 = [g_A][M]^3$$

$$\Rightarrow \boxed{g_A = [M]^1}$$

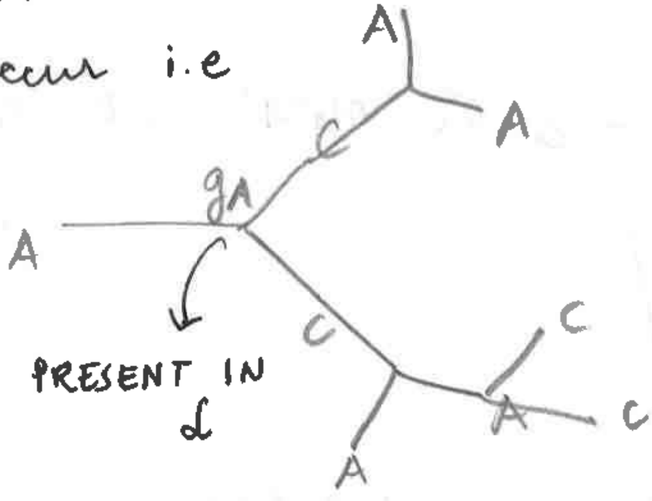
Also, $\boxed{g_B = [M]^1}$

2.2) What forbids CA^2 ?



Trivially, we expect that if $g_C = 0$; it will forbid CA^2 term in $\mathcal{L}$.

On a closer evaluation, we see that presence of CA^2 term will mean that $C \rightarrow A + A$ can occur i.e 2.2



This process will keep on proliferating if $g_c \neq 0$ or CA^2 term is present.

Symmetry of the vacuum state will be lost, probably !!

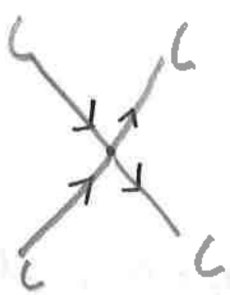
ALSO, is it because of Crossing symmetry $\begin{pmatrix} A \Rightarrow C \\ C \Rightarrow A \end{pmatrix}$?

2.3) What forbids C^4 ?

Firstly, we have three scalar fields A, B, C here.

Presence of C^4 term will mean a self-interacting contribution of C which is not possible. Also, because there is no $\underline{g_c}$.

Also, there cannot be a 4-point vertex here



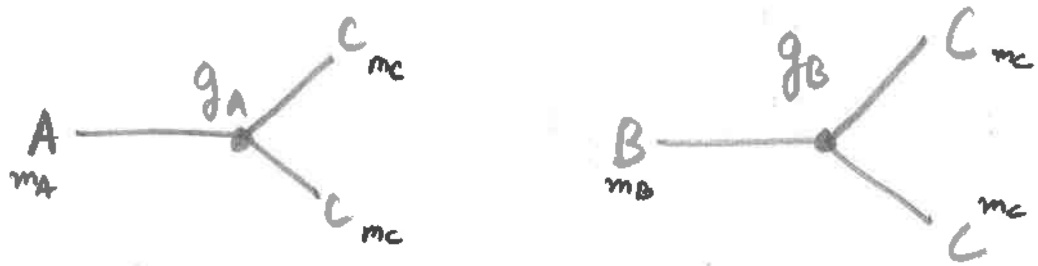
$\longrightarrow$ not allowed

2-3

I also feel that c^4 term will shift the vacuum expectation value i.e change the VEV.

Symmetry of the ground state will be lost because of c^4 term.

2.4)



g_A & g_B can be treated as small if

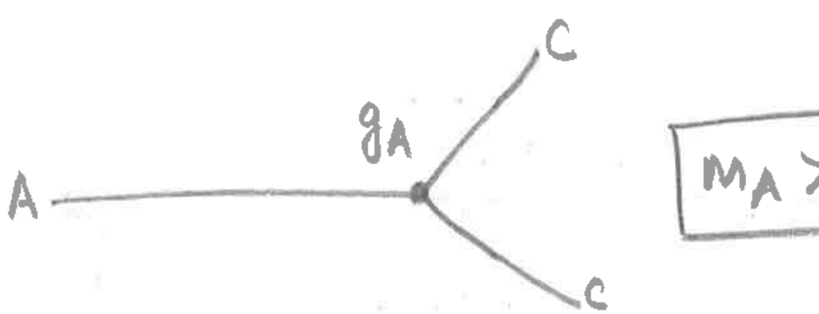
$$g_A, g_B << m_A, m_B, m_C$$

i.e $\boxed{g_A, g_B << m}$ (order of magnitude of fields)

Also, we saw in part (A) that

$[g_A] = [g_B] = [M]^1$ so it makes sense !!

2.5)



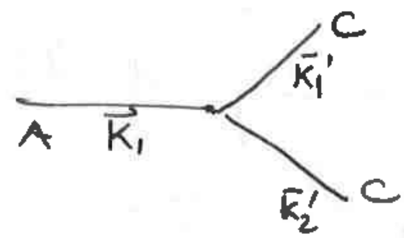
$\boxed{M_A >> 2M_c}$

Tree level for $A \longrightarrow CC$

$\Gamma = \frac{1}{S} \int d\Gamma$ $\qquad$ $A \to C + C$ $\qquad$ 2-4

where $d\Gamma = \frac{1}{S} \frac{1}{2E_1} |\mathcal{T}|^2 \, dLIPS_2(k_1)$

using Srednicki 11.30 for $dLIPS(k)$



$$\Gamma = \frac{1}{S} \int \frac{1}{2E_1} |\mathcal{T}|^2 \frac{|\bar{K}_1'|}{16\pi^2\sqrt{s}} \, d\Omega_{CM}$$

using 11.3 for $|\bar{K}_1'|$, we get

$$= \frac{1}{S} \int \frac{1}{2E_1} |\mathcal{T}|^2 \frac{\frac{1}{2\sqrt{m_A^2}} \sqrt{m_A^4 - 2(m_c^2 + m_c^2) m_A^2 + (m_c^2 - m_c^2)}}{16\pi^2\sqrt{s}} \, d\Omega_{CM}$$

(zero)

$$= \frac{1}{S} \frac{1}{64\pi^2 m_A^3} \int |\mathcal{T}|^2 \sqrt{m_A^4 - 2(2m_c^2) m_A^2} \, d\Omega \quad \left[\text{here } \boxed{E_1^2 = m_A^2}\right.$$

$$= \frac{1}{S} \frac{1}{64\pi^2 m_A^3} m_A^2 \sqrt{1 - \frac{4m_c^2}{m_A^2}} \int |\mathcal{T}|^2 \, d\Omega$$

$\mathcal{T} = -ig_A$ and $S = \frac{1}{2}$ and $d\Omega = 4\pi$

we get,

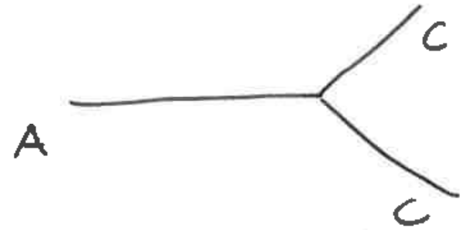
$$\Gamma = \frac{1}{32\pi^2 m_A} \sqrt{1 - \frac{4m_c^2}{m_A^2}}$$

$$= \frac{1}{32\pi^2 m_A} \sqrt{1 - \left(\frac{2m_c}{m_A}\right)^2} \simeq \frac{1}{32\pi^2 m_A}$$

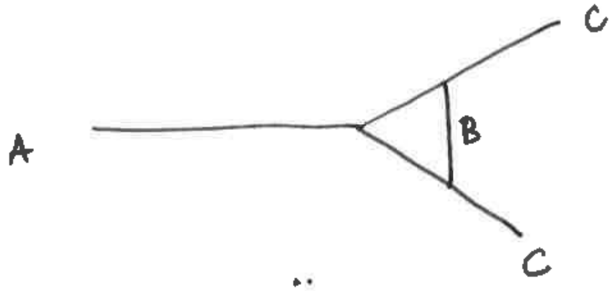
→ equivalent ✓ $2m_c \ll m_A$

✓.

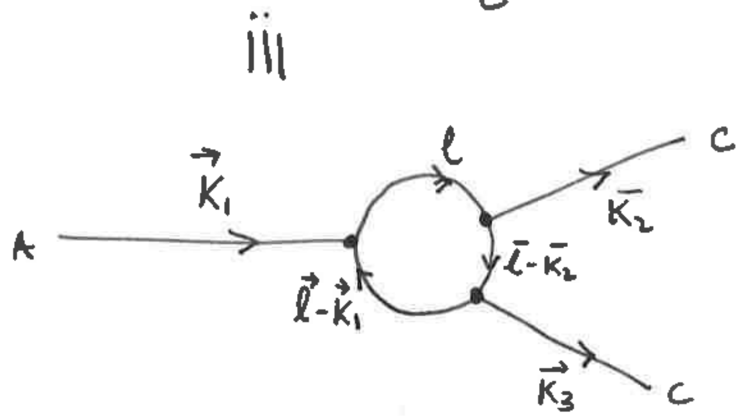
2.6)



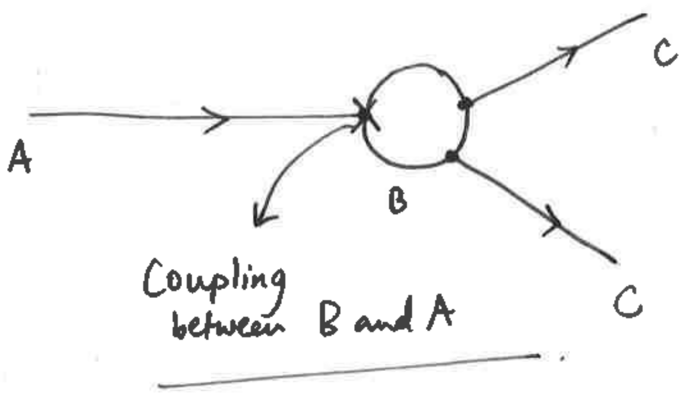
Tree



One loop correction



3 vertex factors in τ



I see two possibilities here:

① Assuming $g_A \simeq g_B \cong g$, we see that three vertex will draw a factor of $(g)^3$

$$\text{But} \quad \Gamma_{\text{one-loop}} \sim |\tau|^2 \sim (g)^6$$

Clearly since we assumed $g_A \simeq g_B \simeq g$ to be very small ; g^6 is almost zero. (2-6)

Hence $\Gamma_{\text{tree level}} >>> \Gamma_{\text{one-loop}}$

$$\frac{\Gamma_{\text{tree-level}}}{\Gamma_{\text{one-loop}}} \gg 1$$

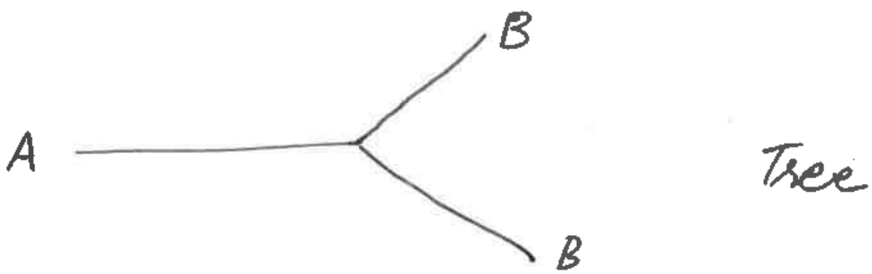
✓ So accuracy is <u>GOOD</u>.

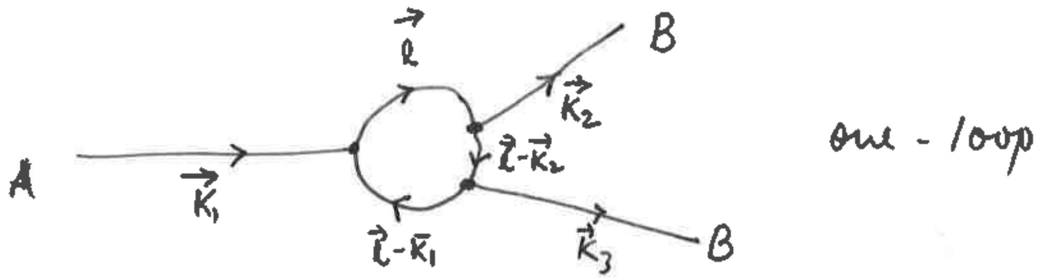
<u>2nd inference</u>:

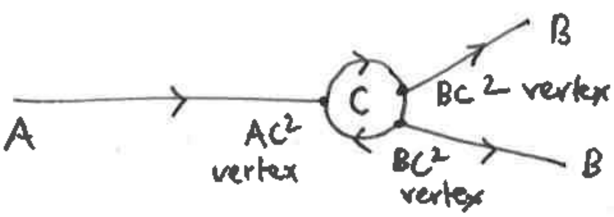
In the given $\mathcal{L}$; we don't have any coupling between A and B of any form i.e AB^2 or BA^2. So. that means one of the vertex factor contribution is <u>ZERO</u>; which makes $\boxed{\Gamma_{\text{one loop}} = 0}$

In Either case, the approximation of tree level is good.

2.7) $A \to BB$







Evaluation of this Feynman diagram tells us that the vertex will have integral like assuming $g_A = g_B \cong g$

$$iV \sim (ig)^3 \left(\frac{1}{i}\right)^3 \int \frac{d^4 l}{(2\pi)^4} \tilde{\Delta}((l-k_1)^2)\, \tilde{\Delta}((l+k_2))\, \tilde{\Delta}(l^2)$$

where $\tilde{\Delta}(k^2) = \dfrac{1}{k^2 + m^2 - i\epsilon}$

Power counting tells us that denominator goes as $\dfrac{1}{l^6}$

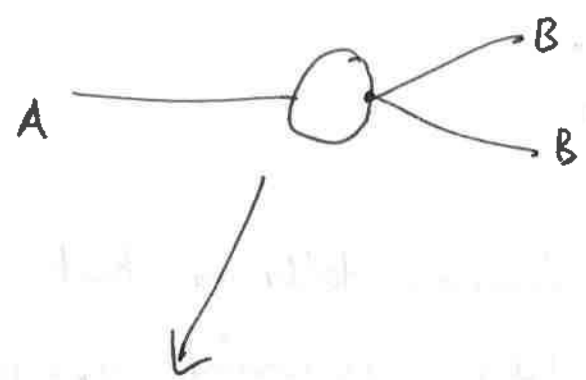
$$\int \frac{d^4 l}{(l)^6}$$ is <u>NOT</u> <u>UV</u> <u>DIVERGENT</u>

$\int \frac{d^4k}{k^4}$ is log - divergent

$\int \frac{d^4k}{k^2}$ is quadratic divergent

$\int \frac{d^4k}{k^3}$ is linear divergent

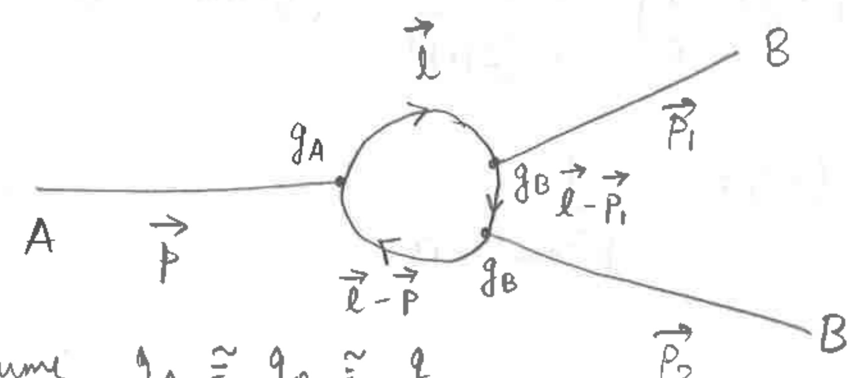
Also, note that diagrams like



are not allowed because $\underline{B^2C^2}$ coupling not possible

Infact this diagram is $\underline{\text{LOG - DIVERGENT}}$!!

2.8)



$\vec{p} = \vec{p}_1 + \vec{p}_2$

Assume $g_A \cong g_B \cong g$

$$\mathcal{M} = \frac{(-ig)^3}{2} \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 - m_c^2 + i\epsilon} \frac{1}{(l-p_1)^2 - m_c^2 + i\epsilon} \cdot \frac{1}{(l-p)^2 - m_c^2 + i\epsilon}$$

Clearly, this integral is UV-finite !!

2.9) Feynman parametrization combines denominators in the loop integrals.

$$\frac{1}{ABC} = \frac{1}{\ell^2 - m_c^2 + i\epsilon} \quad \frac{1}{(\ell-p_1)^2 - m_c^2 + i\epsilon} \quad \frac{1}{(\ell-p)^2 - m_c^2 + i\epsilon}$$

Then

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{2\,\delta(x+y+z-1)}{\left[(\ell^2 - m_c^2 + i\epsilon)\,x + ((\ell-p_1)^2 - m_c^2 + i\epsilon)\,y + ((\ell-p)^2 - m_c^2 + i\epsilon)\,z\right]^3}$$

✓

or

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dz \, \frac{1}{\left[A + (B-A)x + (C-A)z\right]^3}$$

✓✓

2.10) $m_c \gg m_A, m_B$ case

We see that when $m_C \gg m_A, m_B$.

The change in momentum will be negligible when

C emits out B & A i.e

$$|\vec{l}| \cong |\vec{l} - \vec{p}| \cong |\vec{l} - \vec{p}_1|$$

using this in the 1/ABC formula

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{2\delta(x+y+z-1)}{\left[l^2 - m_c^2 + i\epsilon\,[x+y+z]\right]^3}$$

$$= \frac{1}{\left[l^2 - m_c^2 + i\epsilon\right]^3}$$

$x+y+z=1$

$$\mathcal{M} = \frac{(-ig)^3}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - m_c^2 + i\epsilon)^3}$$

$$\cong \cancel{-i^2}\, \frac{ig^3}{2(2\pi)^4} \int \frac{d^4\tilde{l}}{\tilde{l}^6}$$

g^3 is very very small $\ll 1$

$l^2 - m_c^2 = \tilde{l}^2$

$\longrightarrow 0$

Vanishes !!

3) DIRAC SPINORS

$$S \equiv \left(\frac{\vec{p}\cdot\vec{\lambda}}{m_f} \; ; \; \vec{\lambda} + \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E_f)} \right)$$

This is the four-component spin vector. Infact, we can think of this as spin-4 vector.

2 METHODS TO PROVE $S^2 = -1$ i.e $S^\mu S_\mu = -1$

---x

1st method :

$$S^\mu := \left(\frac{\vec{p}\cdot\vec{\lambda}}{m_f} \; ; \; \vec{\lambda} + \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E_f)} \right)$$

$$S_\mu := \left(\frac{\vec{p}\cdot\vec{\lambda}}{m_f} \; ; \; -\vec{\lambda} - \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} \right)$$

Since, Lowering a space index gets a negative sign.

$$S^\mu S_\mu := \left(\frac{(\vec{p}\cdot\vec{\lambda})^2}{m_f^2} + \left[\vec{\lambda} + \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} \right] \left[-\vec{\lambda} - \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} \right] \right.$$

$$= \frac{(\vec{p}\cdot\vec{\lambda})^2}{m_f^2} - 1 - \frac{2(\vec{\lambda}\cdot\vec{p})(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} - \frac{p^2(\vec{p}\cdot\vec{\lambda})^2}{m_f^2(m_f+E)^2} \qquad \vec{\lambda}\cdot\vec{\lambda} = 1$$

$$= (\vec{p}\cdot\vec{\lambda})^2 \left[\frac{1}{m_f^2} - \frac{2}{m_f(m_f+E)} - \frac{p^2}{m_f^2(m_f+E)^2} \right] - 1$$

$$= (\bar{p}\cdot\bar{\lambda})^2\left[\frac{1}{m_f^2} - \frac{2}{m_f(m_f+E)} - \frac{(E+m_f)(E-m_f)}{m_f^2(E+m_f)^2}\right] - 1 \qquad \left\{ p^2 = E^2 - m_f^2 = (E+m_f)(E-m_f) \right.$$

$$= (\bar{p}\cdot\bar{\lambda})^2\left[\frac{1}{m_f^2} - \frac{2}{m_f(m_f+E)} - \frac{(E-m_f)}{m_f^2(E+m_f)}\right] - 1$$

$$= (\bar{p}\cdot\bar{\lambda})^2\left[\frac{E+\cancel{m_f}-2\cancel{m_f}-\cancel{E}+m_f}{m_f^2(E+m_f)}\right] - 1$$

$$= 0 - 1$$

$$= -1 \quad ✓.$$

<u>Second method is more of beauty:</u>

Let's boost to a frame where $\boxed{\vec{p} = 0}$ i.e to a frame where the fermion is at rest. Then

$$S^\mu := (0, \vec{\lambda}) \qquad \text{since } \vec{p} = 0$$

$$S_\mu := (0, -\vec{\lambda}) \qquad \text{Lowering space sign gives } \underline{\text{negative}}$$

$$\Rightarrow S^\mu S_\mu = -\vec{\lambda}\cdot\vec{\lambda}$$

$$= -1 \quad ✓.$$

Now prove $S_\mu p^\mu = 0$.

Here, we will resort to second technique (which is shorter and nicer). Let's boost to a frame where $\vec{p} = 0$

i.e $p^\mu = (m, 0)$

$$S_\mu = \left(\frac{\bar{p}\cdot\bar{\lambda}}{m_f} ; \bar{\lambda} + \frac{\vec{p}\,(\bar{p}\cdot\bar{\lambda})}{m_f(m_f+E)} \right)$$

In frame of moving fermion

$$S_\mu = (0, \vec{\lambda})$$

and $p^\mu = (m, 0)$ since $p^0 = \frac{E}{c} = m$

$c = 1$ here

$$S_\mu p^\mu = 0.m + \bar{\lambda}\cdot 0$$

$$= 0$$

Moving on:

We have the relation,

$$U_f(\bar{p},\bar{\lambda})\,\bar{U}_f(\bar{p},\bar{\lambda}) = \frac{1}{2}(\not{p} + m_f)(1 + \gamma^5 \not{s}) \qquad — (10)$$

using (10) prove

$$\sum_\lambda U_f(\bar{p},\bar{\lambda})\,\bar{U}_f(\bar{p},\bar{\lambda}) = \not{p} + m_f$$

We see that λ (polarization can have two possible orientations). Preferably, let it be left-handed and right-handed.

We write the eq. ⑩ again as,

$$u_f(\bar{p},\bar{\lambda})\,\bar{u}_f(\bar{p},\bar{\lambda}) = (\not{p}+m_f)\left(\frac{1+\gamma^5\gamma^\mu s_\mu}{2}\right)$$ where $\not{s} = \gamma^\mu s_\mu$

Now, infact underlined term is the Right & left projection operator with ⊕ and ⊖ sign for different λ.

Let λ be $\hat{z}$ and $-\hat{z}$ or $\uparrow$ and $\downarrow$

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

s_μ $-s_\mu$ s_μ $-s_\mu$

$$\sum_\lambda u_f(\bar{p},\bar{\lambda})\,\bar{u}_f(\bar{p},\bar{\lambda}) = (\not{p}+m_f)\left(\frac{1+\gamma^5\not{s}}{2}\right) + (\not{p}+m_f)\left(\frac{1-\gamma^5\not{s}}{2}\right)$$

$$= \not{p}+m_f \checkmark$$

Proved.

3.3) Prove

$$\bar{u}_f(\bar{p},\bar{\lambda})\,u_f(\bar{p},\bar{\lambda}) = 2m_f$$

Use cyclic property of Trace i.e

$$\mathrm{Tr}[ABC] = \mathrm{Tr}[BCA] = \mathrm{Tr}[CAB] \checkmark$$

Let's briefly write out some important results :

* $\not{a}\not{b} + \not{b}\not{a} = a_\mu \gamma^\mu b_\nu \gamma^\nu + b_\nu \gamma^\nu a_\mu \gamma^\mu$
$= a_\mu b_\nu \, 2 g^{\mu\nu} = 2\, a\cdot b$ ✓

* $\mathrm{Tr}(\not{a}\not{b}) = \frac{1}{2}\mathrm{Tr}(\not{a}\not{b} + \not{b}\not{a})$
$= \frac{1}{2}\, 2 g^{\mu\nu} a_\mu b_\nu \, \mathrm{Tr}(\mathbb{I})$
$= 4\, a\cdot b$ ✓

* $\mathrm{Tr}(\not{a}_1 \cdots \not{a}_n) = (-1)^n \mathrm{Tr}(\not{a}_1 \cdots \not{a}_n)$
i.e if n = odd, Trace vanishes !!

* $\mathrm{Tr}(\gamma^5 \not{a}\not{b}) = 0$

Note :-
$\mathrm{Tr}[\bar{u}_f u] = \bar{u}_f u$
Since
$\bar{u}_f$ is 1×4 matrix
u_f is 4×1 matrix
hence $\bar{u}_f u$ = 1×1 matrix

Eq. (10) reads,

$$u_f(\vec{p},\vec{\lambda})\, \bar{u}_f(\vec{p},\vec{\lambda}) = \frac{1}{2}(\not{p} + m_f)(1 + \gamma^5 \not{s})$$

$$\Rightarrow \mathrm{Tr}\left[u_f(\vec{p},\vec{\lambda})\, \bar{u}_f(\vec{p},\vec{\lambda})\right] = \frac{1}{2}\mathrm{Tr}\left[\not{p} + \not{p}\gamma^5\not{s} + m_f + m_f\gamma^5\not{s}\right]$$

Applying the cyclic property to L.H.S we get

$$\mathrm{Tr}\left[\bar{u}_f(\vec{p},\vec{\lambda})\, u_f(\vec{p},\vec{\lambda})\right] = \frac{1}{2}\mathrm{Tr}[\not{p}] + \frac{1}{2}\mathrm{Tr}[\not{p}\gamma^5\not{s}] + \frac{1}{2} m_f \mathrm{Tr}\,\mathbb{I} + \frac{m_f}{2}\mathrm{Tr}[\gamma^5\not{s}]$$

$\{\not{p}\gamma^5\not{s} = \gamma_5\not{s}\not{p}\}$

$\Downarrow$

$\bar{u}_f(\vec{p},\vec{\lambda})\, u_f(\vec{p},\vec{\lambda})$

First, second and fourth terms $\longrightarrow 0$
i.e underlined terms

$= 2 m_f$ ✓

Using above results..

3.4) $\overline{u}_f(\vec{p},\vec{\lambda})\,\gamma_5\, u_f(\vec{p},\vec{\lambda})$

Given eq. (10)

$$u_f(\vec{p},\vec{\lambda})\,\overline{u}_f(\vec{p},\vec{\lambda}) = \frac{1}{2}(\not{p}+m_f)(1+\gamma^5\not{s})$$

Premultiply by γ_5 on both sides

$$\gamma_5\, u_f(\vec{p},\vec{\lambda})\,\overline{u}_f(\vec{p},\vec{\lambda}) = \frac{1}{2}\gamma^5(\not{p}+m_f)(1+\gamma^5\not{s})$$

Take trace on both sides and use the cyclic property on L.H.S

$$\mathrm{Tr}\left[\overline{u}_f(\vec{p},\vec{\lambda})\,\gamma_5\, u_f(\vec{p},\vec{\lambda})\right] = \frac{1}{2}\mathrm{Tr}\left[\gamma^5(\not{p}+m_f)(1+\gamma^5\not{s})\right]$$

1X4   4X4   4X1

1X4   4X1

1X1

↓

a number

$$\overline{u}_f(\vec{p},\vec{\lambda})\,\gamma_5\, u_f(\vec{p},\vec{\lambda}) = \frac{1}{2}\mathrm{Tr}\left[\gamma^5\not{p} + \gamma^5\not{p}\gamma^5\not{s} + \gamma^5 m_f + \gamma^5 m_f\gamma^5\not{s}\right]$$

$$= \frac{1}{2}\mathrm{Tr}[\gamma^5\not{p}] + \frac{1}{2}\mathrm{Tr}[\gamma^5\not{p}\gamma^5\not{s}] + \frac{1}{2}\mathrm{Tr}[\gamma^5 m_f] + \frac{1}{2}\mathrm{Tr}[\gamma^5 m_f\gamma^5\not{s}]$$

$$= \frac{1}{2}\mathrm{Tr}[\gamma^5\gamma^\mu p_\mu] + \frac{1}{2}\mathrm{Tr}[-\not{p}\gamma^5\gamma^5\not{s}] + \frac{m_f}{2}\frac{1}{2}\mathrm{Tr}[\gamma^5] + \frac{m_f}{2}\mathrm{Tr}[\gamma^5\gamma^5\not{s}]$$

(first term) ↓ zero

$\{\mathrm{Tr}(A+B) = \mathrm{Tr}(A)+\mathrm{Tr}(B)\}$

$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$

$$= \frac{1}{2}\mathrm{Tr}[\not{s}\not{p}] + \frac{m_f}{2}\frac{1}{2}\mathrm{Tr}[\gamma^5] + \frac{m_f}{2}\mathrm{Tr}[\not{s}]$$

↓ ZERO since $\mathrm{Tr}[\gamma^5]=0$

$$= \frac{1}{2}\mathrm{Tr}[\not{s}\not{p}] + \frac{m_f}{2}\mathrm{Tr}[\not{s}]$$

$$= \frac{1}{2} \mathrm{Tr}[\not{s}\not{p}] + \frac{m_f}{2} \mathrm{Tr}[\not{s}]$$

$\downarrow$ zero $\qquad$ $\underline{\mathrm{Tr}[\gamma^\mu s_\mu] = 0}$

$$= \frac{1}{2} 4 \vec{s}.\vec{p} \qquad \mathrm{Tr}[\not{a}\not{b}] = 4\, a.b$$

$$= 2\, s.p$$

$$= 0 \qquad \text{[since we proved before } s.p = 0\text{]}$$

✓

3.5) $\mathcal{L} = y \bar{\psi}_1 (1-\gamma_5) \psi_2 \phi$

In d, dimensions $[L] = [M]^d$

$$[\phi] = [M]^{\frac{d-2}{2}}$$

$$[\bar{\psi}_1] = [\psi_2] = [M]^{\frac{d-1}{2}}$$

$$[M]^d \simeq y\, [M]^{\frac{d-2}{2} + d-1}$$

$$\simeq y\, [M]^{\frac{d-2+2d-2}{2}}$$

$$[M]^d \simeq y\, [M]^{\frac{3d-4}{2}}$$

$$[y] = [M]^{d - \frac{(3d-4)}{2}}$$

$$\sim [M]^{\frac{2d-3d+4}{2}}$$

$$\sim [M]^{\frac{4-d}{2}}$$

Now, if we have $d = 4$, then $[y] = 0$

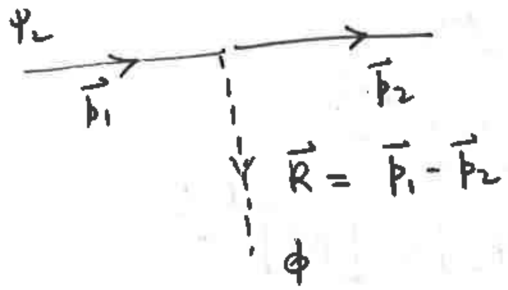
In $D = 4$, y is dimensionless. (3-8)

3.6) $\psi_1 \longrightarrow \phi\,\psi_2$

This decay is not possible since $m_{\psi_2} \gg m_{\psi_1}, m_\phi$

3.7) Decay $\psi_2 \longrightarrow \phi\,\psi_1$

Our lowest order Feynman diagram for this decay:



Following Feynman rules on P-288 (→ Srednicki), we get

$$i\mathcal{T}_{\psi_2 \to \phi + \psi_1} = i\,\bar{u}_{s_2'}(\bar{p}_2')\, y\,(1-\gamma_5)\, u_{s_1}(p_1)\cdot \frac{1}{k^2 - m_\phi^2 - i\epsilon}$$

(↑ not there!! — the factor $\frac{1}{k^2 - m_\phi^2 - i\epsilon}$ is crossed out)

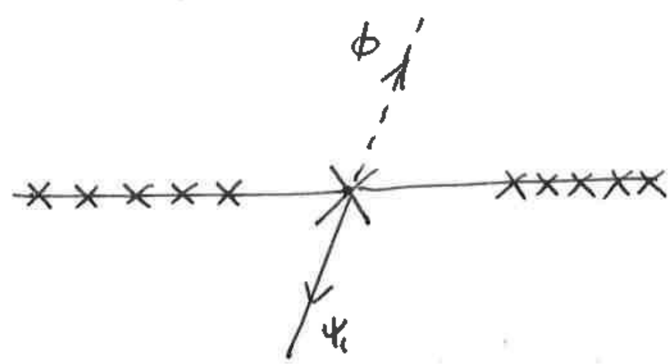
here y = dimensionless ✓.

3.8) Rest frame of ψ_2 (incoming particle).

We can assume $\frac{m_1}{m_2}$ and $\frac{m_\phi}{M_2}$ to be very small.

Basically, the centre of mass (COM) frame will be ψ_2 frame [since other masses are small]

ψ_2 at rest decays into ϕ and ψ_1.



Rest frame of ψ_2.

Clearly $|\bar{p}_i| = |\bar{p}_f|$

$\rightarrow |\vec{p}_\phi| = |\vec{p}_{\psi_1}|$

i.e $\boxed{p_\phi = p'_{2\psi_1}}$

Differential decay rate

$d\Gamma = \frac{1}{2E_1} |\tau|^2 dLIPS_{n'}(k_1)$

Given $\vec{\lambda} \cdot \vec{p}'_{2\psi_1} = |\vec{\lambda}| \, |\vec{p}'_{2\psi_1}| \cos\theta$

$\tau = \mathcal{M} = \bar{U}_{s_2}(\bar{p}'_{2\psi_2}) \, y \, (1-\gamma_5)$

$\tau^* = \bar{\tau} = \bar{V}_{s_1}(p'_1) \, y \, (1+\gamma_5)$

$\langle|\tau|^2\rangle = \sum_{\substack{\text{outgoing} \\ \text{fermion}}} Tr\left[\bar{U}_{s_2}(\bar{p}'_{2\psi_2}) \, y \, (1-\gamma_5) \, V_{\bar{s}_1}(\bar{p}_1) \, y \, (1+\gamma_5)\right]$

$$d\Gamma = \frac{1}{2E_1} |T|^2 \, dLIPS_2(k)$$

here in ψ_2 frame E_1 = mass of ψ_2 $= m_2$

$dLIPS_2(k) = (2\pi)^4 \delta(k_{\psi_p} - k_{\psi_2}) \, d\tilde{k}_1 d\tilde{k}_2'$

$$d\tilde{k} = \frac{d^3k}{(2\pi)^3 2k^0}$$

**

3.9) **Intuition:** Clearly for polarized case, we expect a dependence on the angle 'θ'. We saw that $\vec{\lambda}.\vec{p}_{\psi_1} = |\vec{\lambda}| |\vec{p}_{\psi_1}| \cos\theta$. Hence, we believe that the outcome decay rate will depend on $\underline{\cos^2\theta}$.

Now $\cos^2 0 = \cos^2 \pi = 1$

Hence, at $\theta = 0$ and π, we'll have two things:

1) Decay rate will be <u>maximum</u> <u>OR</u> <u>minimum</u> depending on whether $\underline{\cos^2\theta}$ is in numerator or denominator

2) Value is same for $\theta = 0$ & $\theta = \pi$

We expect that the measurement is same for θ and $\pi - \theta$, due to azimuthal symmetry; which will be obvious!

3.10) Decay rate of unpolarized fermion will not depend on 'θ' because when we sum (average) over all initial possible polarizations we are kind of doing $\langle \cos^2\theta \rangle = \frac{1}{2}$. This eliminates the θ-dependence

Realtempting 3.8

outgoing ψ_1 and ϕ have p' & p''

$$|\mathcal{M}|^2 = \sum_{\substack{\text{outgoing}\\ \text{spin}}} \left| \bar{u}(p') y (1+\gamma^5) u(p) \right|^2$$

$$= \sum_{\text{spins}} \bar{u}(p') y (1+\gamma^5) v(p) \bar{v}(p) y (1+\gamma^5) u(p')$$

$$= Tr\left[\not{p}' y (1+\gamma^5) \not{p} y (1+\gamma^5) \right] \longrightarrow$$

$$=$$

This should go down to $\cos^2\theta$ somehow!!

1

1. $L = \frac{\dot{x}^2}{2} - \frac{\omega^2 x^2}{2} - g x^3 - h x^4$

27/30

$m = 1, \ \hbar = 1$

Sol^n: We know that S (action) and $\hbar$ (reduced Planck's constant) have same units.

$[S] = [\hbar] = 1$ ~~0~~ [here] → ~~usually take both dimensionless~~ ✓

okay, I see your notation now

$\hbar$ = Energy ⊗ Time [Dimension]

$$\Rightarrow [E] = \frac{1}{[t]} \quad \text{— (1)}$$

Now $L \sim \frac{\dot{x}^2}{2} \sim \frac{[x]^2}{[t]^2}$

$$[L\,t] \sim [S] \sim 1 \sim \frac{[x]^2}{[t]} \cong 1$$

$$\Rightarrow x = [t]^{1/2}$$

From (1) we get

$$\boxed{x = [E]^{-1/2}}$$ ✓

$$\boxed{L \sim \frac{[x]^2}{[t]^2} \sim \frac{[E]^{-1}}{[E]^{-2}} \sim [E]}$$

$$\omega^2 x^2 = [E]$$

$$\omega^2 [E]^{-1} = [E]$$

$$\boxed{\omega = [E]} \checkmark$$

$$[g x^3] = [E]$$

$$\boxed{[g] = \frac{[E]}{[x]^3} = \frac{[E]}{[E]^{-3/2}} = [E]^{5/2}} \checkmark$$

$$\boxed{h = \frac{[E]}{[x]^4} = \frac{[E]}{[E]^{-2}} = [E]^3} \checkmark$$

1.2) Clearly, h & g are not dimensionless.

$$[h] = [E]^3 \quad ; \quad [g] = [E]^{5/2}$$

Dimensionless conditions yield;

$$h_{new} = \frac{h_{old}}{\omega^3} \ll 1 \checkmark$$

$$g_{new} = \frac{g_{old}}{\omega^{5/2}} \ll 1 \checkmark$$

So, in this condition, our perturbation theory will be valid.

1.3) $L = \frac{\dot{\varphi}^2}{2} - \frac{\omega^2 \varphi^2}{2} - g\varphi^3 - h\varphi^4$

$x \longmapsto \varphi$

The interaction term has both cube and quartic dependence on φ.

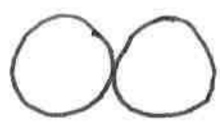
we'll look for diagrams which have no external legs i.e $E = 0$ and construct the required three diagrams.

φ^3 theory $\quad E = 2P - 3V$

if $E = 0$; nothing possible with $V = 1$.

φ^4 theory $\quad E = 2P - 4V$

if $V = 1$, $P = 2$ then $E = 0$



we get other two diagrams due to φ^3 dependence

$P = 3$
$V = 2$
$E = 0$

$P = 3$
$V = 2$
$E = 0$

for UV behavior, look at large k dep. of integrands

They are not UV or IR divergent because we have a mass term, coefficient of φ^2

Mass term does not stop UV divergence

Mass presence $\Longrightarrow$ IR safe ✓

Also, $D = 0+1 = 1$ i.e we have $\frac{dk}{2\pi}$ measure

and loops contribute as $\sim \frac{1}{k^2}$

i.e Power of 'k' in the numerator < Power of 'k' in denominator

$\Rightarrow$ UV Safe. ✓ ah, good. okay!

The scenario will change if we consider $D = 4$.

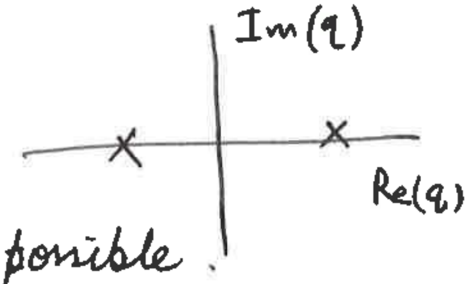
This is valid here since $D = 1$ [UV & IR safe!!]

✓

1.4) In the limit $\omega \to 0$. we are spoiling the mass term in $\mathcal{L}$. It will result in no mass (massless) $\Longrightarrow$ IR divergent. ✓

Also, intuitively $\omega \to 0$ means $k \to 0$ i.e $\lambda \to \infty$, a look at EM spectrum tells ?? that it should be IR divergent (hence the name).. Not sure what you mean...

1.5) In the $\lim_{\omega\to 0}$, the poles are hitting the real 'q' plane and Wick rotating seems not-possible.



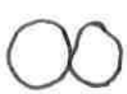
Remedy: Add $i\epsilon$ and save the poles to be on real 'q' axis

$$\frac{1}{k^2-\omega^2} \xrightarrow{\lim_{\omega\to 0}} \frac{1}{k^2-i\epsilon}$$

We therefore regulate the integral by adding small 'ϵ' which can be later put to zero.

okay, but real issue is perturbativity. You will be violating conditions in part 2!

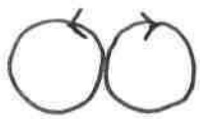
1.6) Let's go back to the diagrams we had for the problem

The problem	Symmetry factor 'S'
	$\frac{1}{8}$
	$\frac{1}{8}$
	$\frac{1}{12}$

✓ } Recall previous assignments.

Propagator $\dfrac{i}{k^2-\omega^2}$

Loop integrals $\dfrac{dk}{2\pi}$



There's no internal line here connecting the two loop so, we'll calculate due to one of them and square it.

Symmetry factor (the $\frac{1}{8}$) — Since we have h not $\frac{h}{4!}$ (the 4!)

$$i\mathcal{M} = \frac{1}{8} \times 4! \; i \int \frac{dk}{(2\pi)} \frac{i}{k^2-\omega^2+i\epsilon}$$

NOTE: The factor of 4! here is just because couplings are without factor of n! like QFT we multiply to take this into account. In the next two diagrams, we'll have to multiply by 3! similarly !!

$$iM = \frac{1}{8}\, 4!\, \frac{(i)^2 \hbar}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{k^2 - \omega^2 + i\epsilon}$$

Let's Wick rotate this now i.e

$$k = i\tilde{k}$$

$$k^2 = -\tilde{k}^2$$

$$dk \longrightarrow i\, d\tilde{k}$$

$$iM = \frac{1}{8}\, 4!\, \frac{(i^3)\hbar}{(2\pi)^2} \left[\int \frac{d\tilde{k}}{-\tilde{k}^2 - \omega^2} \right]^2 \qquad [\epsilon \to 0]$$

$$= \frac{1}{8}\, 4!\, \frac{-i\hbar}{(2\pi)^2} \left[\int \frac{d\tilde{k}}{-\tilde{k}^2 - \omega^2} \right]^2$$

$$= \frac{4 \times 3 \times 2}{8\ 4} \frac{-i\hbar}{4\pi^2} \left(\frac{\pi}{\omega} \right)^2 \qquad [\text{we used the given integral}]$$

$$= -i\, \frac{3}{4} \frac{\hbar}{\omega^2}$$

$$M = \frac{-3}{4} \frac{\hbar}{\omega^2} \checkmark$$

But

$$\Delta E = -M = \frac{3}{4} \frac{\hbar}{\omega^2}$$

Call later

First chunck of ΔE

Now, we move to the second diagram

$$S = \frac{1}{8}$$

Now, we have $P = 3$; $V = 2$

$$i\mathcal{M} = \frac{1}{8} \times (3!)^2 \, i^2 g^2 \int \frac{dk_1}{(2\pi)} \frac{dk_2}{(2\pi)} \frac{i}{k_1^2 - \omega^2} \frac{i}{k_2^2 - \omega^2} \frac{i}{-\omega^2}$$

For the internal line, we have $k = 0$, since conservation of momentum means k_1 going to left loop and k_2 going to right one.

$$= +\frac{1}{8} \times \frac{\cancel{6}^{3} \times \cancel{6}^{3}}{\cancel{2}\cancel{4}\pi^2 \omega^2} g^2 \, i \left(\frac{\pi}{\omega}\right)^2$$

Integrating over k_1 and k_2 & using the given result

$$= +i \frac{9}{8} \frac{g^2}{\cancel{\pi^2}} \frac{\cancel{\pi^2}}{\omega^4}$$

$$\boxed{i\mathcal{M} = i \frac{9 g^2}{8 \omega^4}} \rightarrow \boxed{\mathcal{M} = + \frac{9}{8} \frac{g^2}{\omega^4}}$$

use later ✓

Now, the second diagram for φ^3 dependence i.e 3rd diagram in overall evaluating is  with $S = \frac{1}{12}$

For  we have $V = 2$; $P = 3$

Note that the middle propagator depends on two entering propagator with $\vec{k}_1$ and $\vec{k}_2$. We'll pull off a δ-$f^{\underline{n}}$ with conservation of momentum at vertices with factor of $(2\pi)^d \, \delta(k_1 + k_2 - k_3)$; but since $\underline{d = 1}$; we get $\underline{(2\pi)\, \delta(k_1 + k_2 - k_3)}$ ..

$$iM = \frac{1}{12} \times \overset{3!}{6} \times \overset{3!}{6} \, (ig)^2 \frac{i^3}{(2\pi)^3} \cdot 2\pi \int dk_1 \, dk_2 \frac{1}{k_1^2 - \omega^2} \frac{1}{k_2^2 - \omega^2} \frac{1}{(k_1 + k_2)^2 - \omega^2}$$

we have already put $\vec{k}_3 = \vec{k}_2 + \vec{k}_1$ in ↗

$$= \frac{6 \times 6}{12} (ig)^2 \frac{i^3}{(2\pi)^2} \int dk_1 \, dk_2 \frac{1}{k_1^2 - \omega^2} \frac{1}{k_2^2 - \omega^2} \frac{1}{(k_1 + k_2)^2 - \omega^2}$$

Again do the Wick rotation and get the value of integral from the given listed integral, we get

$$= \frac{\cancel{6} \times \cancel{6}^{\,3}}{\cancel{12}} \frac{i^2 g^2 i^3}{(2\pi)^2} \left(\frac{\pi^2}{3\omega^4} \right)$$

$$= \frac{3}{12} \, i \, \frac{g^2}{\pi^2} \frac{\pi^2}{\omega^4} = i \frac{3}{12} \frac{g^2}{\omega^4}$$

Now this is

$$iM = i\frac{3}{2}\frac{g^2}{\omega^4}$$

$$\Rightarrow \boxed{M = \frac{3}{12}\frac{g^2}{\omega^4}}$$

Now we add M's from  & 

$$M = \frac{3g^2}{12\omega^4} + \frac{9}{8}\frac{g^2}{\omega^4}$$

$$= \frac{6g^2 + 27g^2}{24\omega^4}$$

$$= \frac{\overset{11}{\cancel{33}}g^2}{\underset{8}{\cancel{24}}\omega^4}$$

$$= \frac{11}{8}\frac{g^2}{\omega^4}$$

But $\Delta E = -M$

$$= \frac{-11}{8}\frac{g^2}{\omega^4}$$

[Contribution from ~~quartic~~ cubic dependence interaction]

$$\Delta E_{total} = \frac{-11}{8}\frac{g^2}{\omega^4} + \frac{3}{4}\frac{h}{\omega^2}$$

☺

PROBLEM 2 : SCALARS

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$$

$$\mathcal{L}_{free} = \frac{1}{2}\left[\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + \partial_\mu C \partial^\mu C - m_A^2 A^2 - m_B^2 B^2 - m_C^2 C^2\right]$$

$$-\mathcal{L}_{int} = \frac{1}{2}\left[g_A A C^2 + g_B B C^2\right]$$

1) Clearly $[\mathcal{L}] = [M]^4$ we are in $d=4$ here ..

$[A] = [B] = [C] = [M]^1$ the scalar fields have mass dimension 1.

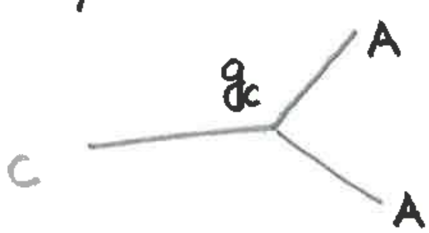
$$\mathcal{L} = [M]^4 = [g_A][M]^3$$

$$\Rightarrow \boxed{g_A = [M]^1}$$

✓

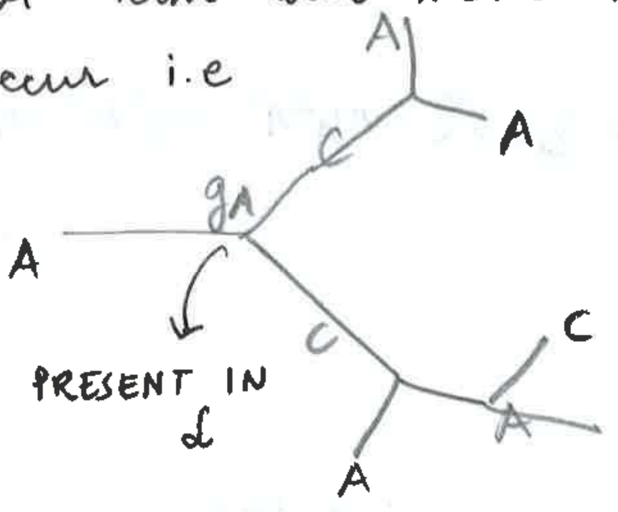
Also, $\boxed{g_B = [M]^1}$

2.2) What forbids CA^2 ?



Trivially, we expect that if $g_c = 0$; it will forbid CA^2 term in $\mathcal{L}$.

On a closer evaluation, we see that presence of CA^2 term will mean that $C \rightarrow A + A$ can occur i.e 2.2



This process will keep on proliferating if $g_c \neq 0$ or CA^2 term is present.

X looking for a symmetry x-form e.g. $C \Leftrightarrow -C$

Symmetry of the vacuum state will be lost, probably !.!

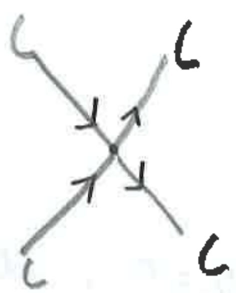
ALSO, is it because of Crossing symmetry $\begin{pmatrix} A \Rightarrow C \\ C \Rightarrow A \end{pmatrix}$?

2.3) What forbids C^4 ?

Firstly, we have three scalar fields A, B, C here.

Presence of C^4 term will mean a self-interacting contribution of C which is not possible. Also, because there is no $\underline{g_c}$.

Also, there cannot be a 4-point vertex here



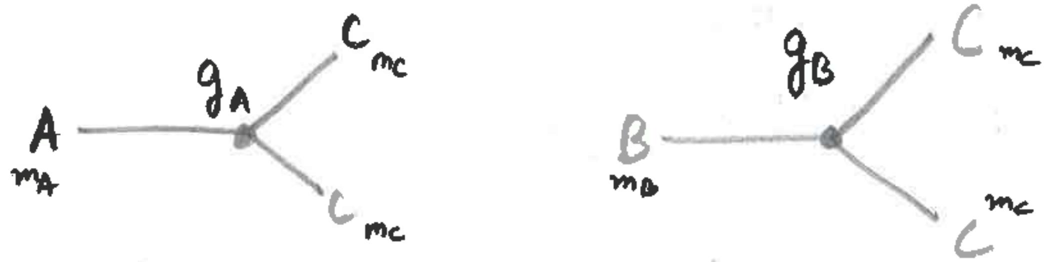
$\longrightarrow$ not allowed

I also feel that c^4 term will shift the vacuum expectation value i.e change the VEV.

Symmetry of the ground state will be lost because of c^4 term.

Hard to come up with a symmetry here... ~~could have~~ $c \Rightarrow$ not really anything available for real scalar c^2 means c^4 allowed!

2.4)



g_A & g_B can be treated as small if

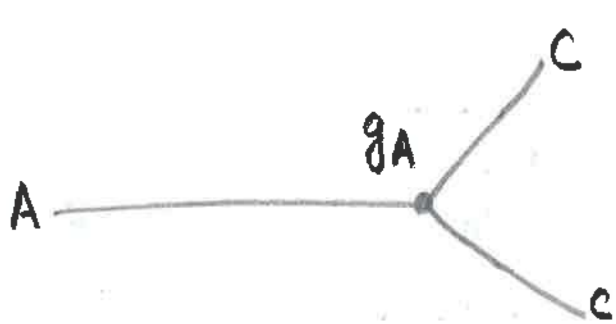
$$g_A, g_B << m_A, m_B, m_C$$

i.e $\boxed{g_A, g_B << m}$ (order of magnitude of fields) ✓

Also, we saw in part (A) that

$[g_A] = [g_B] = [M]^1$ so it makes sense !!

2.5)



$\boxed{M_A >> 2M_C}$

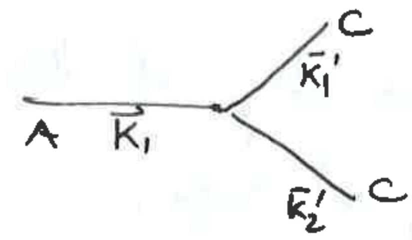
just $m_A > 2m_C$ to be kinematically allowed

Tree level for $A \longrightarrow CC$

$$\Gamma = \frac{1}{S}\int d\Gamma \qquad A \to C + C$$

where $d\Gamma = \frac{1}{S}\,\frac{1}{2E_1}\,|\mathcal{T}|^2\, dLIPS_2(k_1)$

using Srednicki 11.30 for $dLIPS(k)$



$$\Gamma = \frac{1}{S}\int \frac{1}{2E_1}\,|\mathcal{T}|^2\,\frac{|\bar{K}_1'|}{16\pi^2\sqrt{s}}\,d\Omega_{CM}$$

using 11.3 for $|\bar{K}_1'|$, we get

$$= \frac{1}{S}\int \frac{1}{2E_1}\,|\mathcal{T}|^2\,\frac{\frac{1}{2\sqrt{m_A^2}}\sqrt{m_A^4 - 2(m_c^2+m_c^2)m_A^2 + (m_c^2 - m_c^2)^2}}{16\pi^2\sqrt{s}}\,d\Omega_{CM}$$

(arrow from $(m_c^2 - m_c^2)$ term: zero)

$$= \frac{1}{S}\,\frac{1}{64\pi^2 m_A^3}\int |\mathcal{T}|^2\sqrt{m_A^4 - 2(2m_c^2)m_A^2}\,d\Omega \quad \left[\text{here } E_1^2 = m_A^2\right]$$

$$= \frac{1}{S}\,\frac{1}{64\pi^2 m_A^3}\,m_A^2\sqrt{1 - \frac{4m_c^2}{m_A^2}}\int |\mathcal{T}|^2\,d\Omega$$

$\mathcal{T} = -ig_A$ and $S = \frac{1}{2}$ and $d\Omega = 4\pi$ ✓

we get,

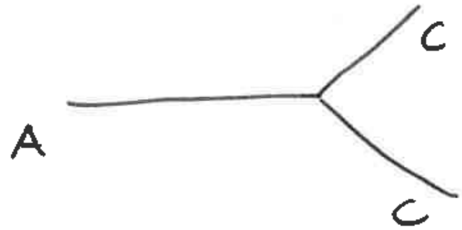
okay, just didn't plug in correctly

$$\Gamma = \frac{g_A^2}{32\pi^2 m_A}\sqrt{1 - \frac{4m_c^2}{m_A^2}}$$

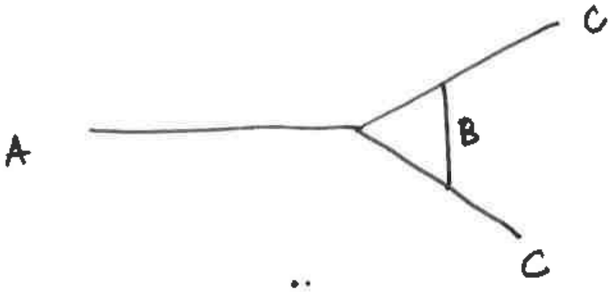
$$= \frac{g_A^2}{32\pi^2 m_A}\sqrt{1 - \left(\frac{2m_c}{m_A}\right)^2} \simeq \frac{1}{32\pi^2 m_A} \quad ✓ \quad 2m_c \ll m_A$$

(arrow: equivalent) ✓

2.6)

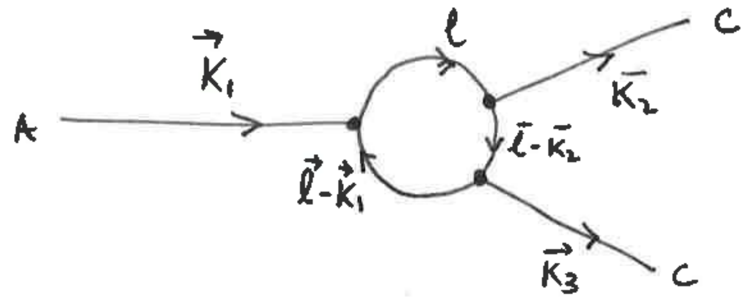


Tree

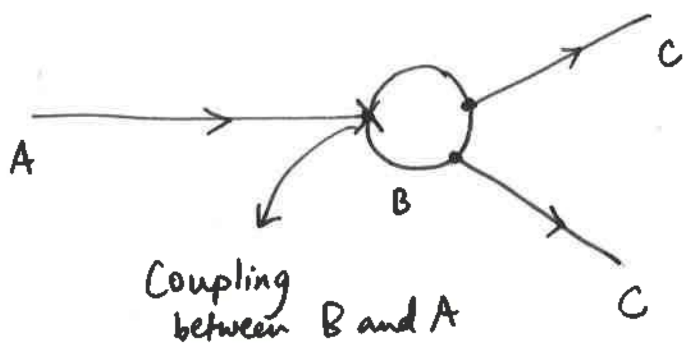


One loop correction

⇓



3 vertex factors in $\mathcal{T}$



not in $\mathcal{L}$ ...

I see two possibilities here:

① Assuming $g_A \cong g_B \cong g$, we see that three vertex will draw a factor of $(g)^3$ i.e. $M_{tree} M^{+}_{loop}$ + h.c.

But $\Gamma_{one-loop} \sim |\mathcal{T}|^2 \sim (g)^6$ NO be careful. Since Γ_{tree} exists, correction is g^4

Clearly since we assumed $g_A \simeq g_B \simeq g$ to be very small ; g^6 is almost zero.

(2-6)

Hence $\Gamma_{\text{tree level}} \ggg \Gamma_{\text{one-loop}}$

$$\frac{\Gamma_{\text{tree-level}}}{\Gamma_{\text{one-loop}}} \gg 1$$

✓ So accuracy is GOOD.

→ Should estimate loop correction to amplitude + compute ratio:

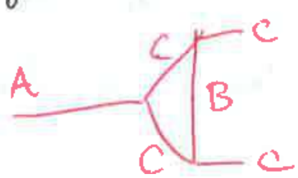
$$\frac{g_A^2}{16\pi^2 M_A^2}$$

(g_A^2 → vertices; M_A^2 → propagators; "loop factor")

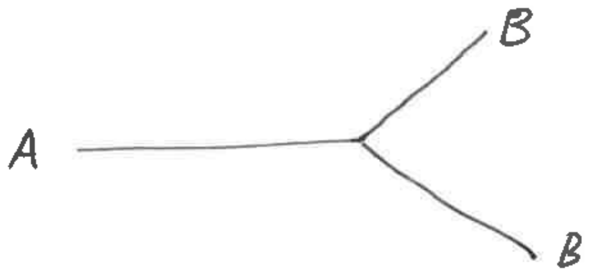
2nd inference :

In the given $\mathcal{L}$; we don't have any coupling between A and B of any form i.e AB^2 or BA^2. So. that means one of the vertex factor contribution is ZERO, which makes $\boxed{\Gamma_{\text{one loop}} = 0}$

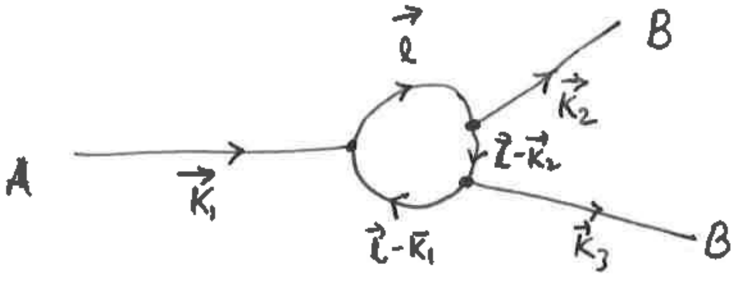
In Either case, the approximation of tree level is good.

have ACC + BCC ⇒ 

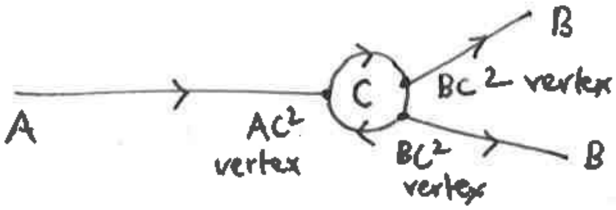
2.7) $A \to BB$



Tree



one - loop



Evaluation of this Feynman diagram tells us that the vertex will have integral like assuming $g_A = g_B \cong g$

$$iV \sim (ig)^3 \left(\frac{1}{i}\right)^3 \int \frac{d^4 l}{(2\pi)^4} \tilde{\Delta}((l-k_1)^2)\, \tilde{\Delta}((l+k_2))\, \tilde{\Delta}(l^2)$$

where $\tilde{\Delta}(k^2) = \dfrac{1}{k^2 + m^2 - i\epsilon}$

Power counting tells us that denominator goes as $\dfrac{1}{l^6}$

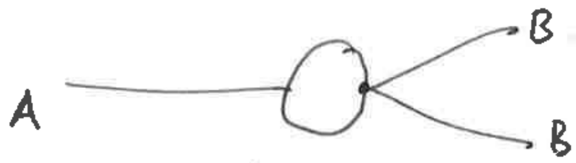
$$\int \frac{d^4 l}{(l)^6}$$ is <u>NOT</u> <u>UV</u> <u>DIVERGENT</u>

$\int \frac{d^4k}{k^4}$ is log - divergent

$\int \frac{d^4k}{k^2}$ is quadratic divergent

$\int \frac{d^4k}{k^3}$ is linear divergent

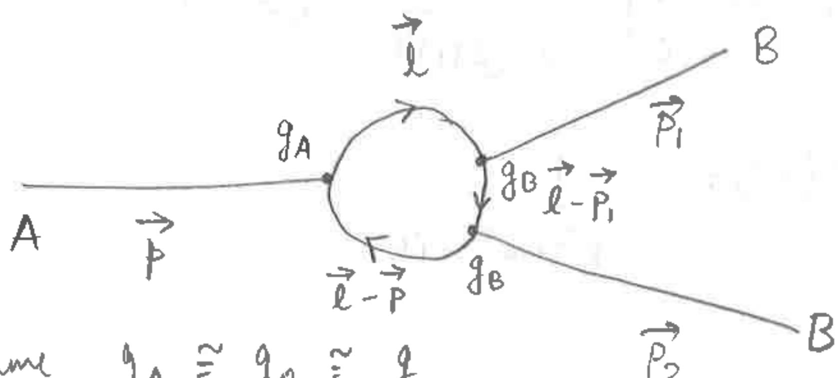
Also, note that diagrams like



are not allowed because $\underline{B^2C^2}$ coupling not possible

Infact this diagram is $\underline{\text{LOG - DIVERGENT}}$!!

2.8)



$\vec{P} = \vec{P_1} + \vec{P_2}$

Assume $g_A \simeq g_B \simeq g$

$$\mathcal{M} = \frac{(-ig)^3}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 - m_c^2 + i\epsilon} \frac{1}{(l-p_1)^2 - m_c^2 + i\epsilon} \cdot \frac{1}{(l-p)^2 - m_c^2 + i\epsilon}$$

✓

$\to p_2$?

Clearly, this integral is UV-finite !!

✓

2.9) Feynman parametrization combines denominators in the loop integrals.

$$\frac{1}{ABC} = \frac{1}{\ell^2 - m_c^2 + i\epsilon} \; \frac{1}{(\ell-p_1)^2 - m_c^2 + i\epsilon} \; \frac{1}{(\ell-p)^2 - m_c^2 + i\epsilon}$$

Then

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \; \frac{2\,\delta(x+y+z-1)}{\left[(\ell^2 - m_c^2 + i\epsilon)x + ((\ell-p_1)^2 - m_c^2 + i\epsilon)y + ((\ell-p)^2 - m_c^2 + i\epsilon)z\right]^3}$$

✓

or

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dz \; \frac{1}{\left[A + (B-A)x + (C-A)z\right]^3}$$

✓

would also be good to see parametrization after symmetrizing integrand (momentum shift, etc.)

2.10) $m_c \gg m_A, m_B$ case

we see that when $m_C \gg m_A, m_B$.

The change in momentum will be negligible when

C emits out B & A i.e

$$|\vec{l}| \simeq |\vec{l} - \vec{p}| \simeq |\vec{l} - \vec{p}_1|$$

using this in the $1/ABC$ formula

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{2\delta(x+y+z-1)}{[l^2 - m_c^2 + i\epsilon[x+y+z]]^3}$$

$$= \frac{1}{[l^2 - m_c^2 + i\epsilon]^3}$$

$x+y+z=1$

$$\mathcal{M} = \frac{(-ig)^3}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - m_c^2 + i\epsilon)^3}$$

$$\simeq \cancel{-i^2} \frac{ig^3}{2(2\pi)^4} \int \frac{d^4\tilde{l}}{\tilde{l}^6}$$

g^3 is very very small $\ll 1$

$l^2 - m_c^2 = \tilde{l}^2$

$\longrightarrow 0$

Vanishes !!

But what is leading behavior?

3) DIRAC SPINORS

$$S \equiv \left(\frac{\vec{p}\cdot\vec{\lambda}}{m_f} \; ; \; \vec{\lambda} + \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E_f)} \right)$$

This is the four-component spin vector. Infact, we can think of this as spin-4 vector.

2 METHODS TO PROVE $S^2 = -1$ i.e $S^\mu S_\mu = -1$

———————— x

1st method :

$$S^\mu := \left(\frac{\vec{p}\cdot\vec{\lambda}}{m_f} \; ; \; \vec{\lambda} + \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E_f)} \right)$$

$$S_\mu := \left(\frac{\vec{p}\cdot\vec{\lambda}}{m_f} \; ; \; -\vec{\lambda} - \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} \right)$$

Since, Lowering a space index gets a negative sign.

$$S^\mu S_\mu := \left(\frac{(\vec{p}\cdot\vec{\lambda})^2}{m_f^2} + \left[\vec{\lambda} + \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} \right] \left[-\vec{\lambda} - \frac{\vec{p}(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} \right] \right.$$

$$= \frac{(\vec{p}\cdot\vec{\lambda})^2}{m_f^2} - 1 - \frac{2(\vec{\lambda}\cdot\vec{p})(\vec{p}\cdot\vec{\lambda})}{m_f(m_f+E)} - \frac{p^2(\vec{p}\cdot\vec{\lambda})^2}{m_f^2(m_f+E)^2} \qquad \Big| \; \vec{\lambda}\cdot\vec{\lambda} = 1$$

$$= (\vec{p}\cdot\vec{\lambda})^2 \left[\frac{1}{m_f^2} - \frac{2}{m_f(m_f+E)} - \frac{p^2}{m_f^2(m_f+E)^2} \right] - 1$$

(3-2)

$$= (\bar{p}\cdot\bar{\lambda})^2\left[\frac{1}{m_f^2} - \frac{2}{m_f(m_f+E)} - \frac{(E+m_f)(E-m_f)}{m_f^2(E+m_f)^2}\right] - 1 \qquad \left\{ \begin{aligned} p^2 &= E^2 - m_f^2 \\ &= (E+m_f)(E-m_f) \end{aligned} \right.$$

$$= (\bar{p}\cdot\bar{\lambda})^2\left[\frac{1}{m_f^2} - \frac{2}{m_f(m_f+E)} - \frac{(E-m_f)}{m_f^2(E+m_f)}\right] - 1$$

$$= (\bar{p}\cdot\bar{\lambda})^2\left[\frac{E + m_f - 2m_f - E + m_f}{m_f^2(E+m_f)}\right] - 1$$

$$= 0 - 1$$

$$= -1 \checkmark .$$

Second method is more of beauty:

Let's boost to a frame where $\boxed{\vec{p} = 0}$ i.e to a frame where the fermion is at rest. Then

$$S^\mu := (0, \vec{\lambda}) \qquad \text{since } \vec{p} = 0$$

$$S_\mu := (0, -\vec{\lambda}) \qquad \text{Lowering space sign gives negative}$$

$$\Rightarrow S^\mu S_\mu = -\vec{\lambda}\cdot\vec{\lambda}$$

$$= -1 \checkmark .$$

Now prove $S_\mu p^\mu = 0$.

Here, we will resort to second technique (which is shorter and nicer). Let's boost to a frame where $\vec{p} = 0$

i.e $\boxed{p^\mu = (m, 0)}$

$$S_\mu = \left(\frac{\bar{p}\cdot\bar{\lambda}}{m_f} ; \bar{\lambda} + \frac{\bar{p}(\bar{p}\cdot\bar{\lambda})}{m_f(m_f+E)} \right)$$

In frame of moving fermion

$$S_\mu = (0, \vec{\lambda})$$

and $p^\mu = (m, 0)$ since $\boxed{p^0 = \frac{E}{c} = m}$

c = 1 here

$$S_\mu p^\mu = 0.m + \bar{\lambda}\cdot 0$$
$$= 0$$

Moving on:

We have the relation,

$$U_f(\bar{p},\bar{\lambda})\,\bar{U}_f(\bar{p},\bar{\lambda}) = \frac{1}{2}(\not{p} + m_f)(1 + \gamma^5 \not{s}) \quad — (10)$$

using (10) prove

$$\sum_\lambda U_f(\bar{p},\bar{\lambda})\,\bar{U}_f(\bar{p},\bar{\lambda}) = \not{p} + m_f$$

We see that λ (polarization can have two possible orientations). Preferably, let it be left-handed and right-handed.

We write the eq. (10) again as,

$$u_f(\bar{p},\bar{\lambda})\,\bar{u}_f(\bar{p},\bar{\lambda}) = (\not{p}+m_f)\left(\frac{1+\gamma^5\gamma^\mu s_\mu}{2}\right)$$ where $\not{s} = \gamma^\mu s_\mu$

Now, infact underlined term is the Right & left projection operator with ⊕ and ⊖ sign for different λ.

Let λ be $\hat{z}$ and $-\hat{z}$ or ↑ and ↓

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

s_μ $-s_\mu$ s_μ $-s_\mu$

$$\sum_\lambda u_f(\bar{p},\bar{\lambda})\,\bar{u}_f(\bar{p},\bar{\lambda}) = (\not{p}+m_f)\left(\frac{1+\gamma^5\not{s}}{2}\right) + (\not{p}+m_f)\left(\frac{1-\gamma^5\not{s}}{2}\right)$$

$$= \not{p}+m_f$$ ✓

Proved.

3.3) Prove

$$\bar{u}_f(\bar{p},\bar{\lambda})\,u_f(\bar{p},\bar{\lambda}) = 2m_f$$

Use cyclic property of Trace i.e

$$Tr[ABC] = Tr[BCA] = Tr[CAB]$$ ✓.

Let's briefly write out some important results :

* $\not{a}\not{b} + \not{b}\not{a} = a_\mu \gamma^\mu b_\nu \gamma^\nu + b_\nu \gamma^\nu a_\mu \gamma^\mu$
$= a_\mu b_\nu 2 g^{\mu\nu} = 2\, a \cdot b$ ✓

* $Tr(\not{a}\not{b}) = \frac{1}{2} Tr(\not{a}\not{b} + \not{b}\not{a})$
$= \frac{1}{2} 2 g^{\mu\nu} a_\mu b_\nu \, Tr(\mathbb{I})$
$= 4\, a \cdot b$ ✓

* $Tr(\not{a}_1 \ldots \not{a}_n) = (-1)^n Tr(\not{a}_1 \ldots \not{a}_n)$
i.e if n = odd, Trace vanishes !!

* $Tr(\gamma^5 \not{a}\not{b}) = 0$

Note :-
$Tr[\bar{u}_f u] = \bar{u}_f u$
Since
$\bar{u}_f$ is 1×4 matrix
u_f is 4×1 matrix
hence $\bar{u}_f u$ = 1×1 matrix

———×———

Eq. (10) reads,

$$u_f(\vec{p},\vec{\lambda})\, \bar{u}_f(\vec{p},\vec{\lambda}) = \frac{1}{2}(\not{p} + m_f)(1 + \gamma^5 \not{s})$$

$$\Rightarrow Tr[u_f(\vec{p},\vec{\lambda})\, \bar{u}_f(\vec{p},\vec{\lambda})] = \frac{1}{2} Tr[\not{p} + \not{p}\gamma^5\not{s} + m_f + m_f \gamma^5 \not{s}]$$

Applying the cyclic property to L.H.S we get

$$Tr[\bar{u}_f(\vec{p},\vec{\lambda})\, u_f(\vec{p},\vec{\lambda})] = \underline{\frac{1}{2} Tr[\not{p}]} + \underline{\frac{1}{2} Tr[\not{p}\gamma^5\not{s}]} + \frac{1}{2} m_f Tr\,\mathbb{I} + \underline{\frac{m_f}{2} Tr[\gamma^5 \not{s}]}$$

$\left\{ \not{p}\gamma^5\not{s} = \gamma_5 \not{s}\not{p} \right\}$

⇓

$\bar{u}_f(\vec{p},\vec{\lambda})\, u_f(\vec{p},\vec{\lambda})$

First, second and fourth terms → 0
i.e underlined terms

$= 2 m_f$ ✓

Using above results..

3.4) $\overline{u}_f(\vec{p},\vec{\lambda})\gamma_5 u_f(\vec{p},\vec{\lambda})$ (3-6)

Given eq. (10)

$$u_f(\vec{p},\vec{\lambda})\,\bar{u}_f(\vec{p},\vec{\lambda}) = \frac{1}{2}(\not{p}+m_f)(1+\gamma^5\not{s})$$

Premultiply by γ_5 on both sides

$$\gamma_5 u_f(\vec{p},\vec{\lambda})\,\bar{u}_f(\vec{p},\vec{\lambda}) = \frac{1}{2}\gamma^5(\not{p}+m_f)(1+\gamma^5\not{s})$$

Take trace on both sides and use the cyclic property on L.H.S

$$\mathrm{Tr}\left[\bar{u}_f(\vec{p},\vec{\lambda})\gamma_5 u_f(\vec{p},\vec{\lambda})\right] = \frac{1}{2}\mathrm{Tr}\left[\gamma^5(\not{p}+m_f)(1+\gamma^5\not{s})\right]$$

1X4 4X4 4X1

1x4 4x1

1x1

↓

a number

$$\bar{u}_f(\vec{p},\vec{\lambda})\gamma_5 u_f(\vec{p},\vec{\lambda}) = \frac{1}{2}\mathrm{Tr}\left[\gamma^5\not{p} + \gamma^5\not{p}\gamma^5\not{s} + \gamma^5 m_f + \gamma^5 m_f\gamma^5\not{s}\right]$$

$$= \frac{1}{2}\mathrm{Tr}[\gamma^5\not{p}] + \frac{1}{2}\mathrm{Tr}[\gamma^5\not{p}\gamma^5\not{s}] + \frac{1}{2}\mathrm{Tr}[\gamma^5 m_f] + \frac{1}{2}\mathrm{Tr}[\gamma^5 m_f\gamma^5\not{s}]$$

$$= \frac{1}{2}\mathrm{Tr}[\gamma^5\gamma^\mu p_\mu] + \frac{1}{2}\mathrm{Tr}[-\not{p}\gamma^5\gamma^5\not{s}] + \frac{m_f}{2}\frac{1}{2}\mathrm{Tr}[\gamma^5] + \frac{m_f}{2}\mathrm{Tr}[\gamma^5\gamma^5\not{s}]$$

(first term ↓ zero)

$\{\mathrm{Tr}(A+B) = \mathrm{Tr}(A)+\mathrm{Tr}(B)\}$

$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$

$$= \frac{1}{2}\mathrm{Tr}[\not{s}\not{p}] + \frac{m_f}{2}\frac{1}{2}\mathrm{Tr}[\gamma^5] + \frac{m_f}{2}\mathrm{Tr}[\not{s}]$$

↓ ZERO since $\mathrm{Tr}[\gamma^5]=0$

$$= \frac{1}{2}\mathrm{Tr}[\not{s}\not{p}] + \frac{m_f}{2}\mathrm{Tr}[\not{s}]$$

$$= \frac{1}{2}\mathrm{Tr}[\not{s}\not{p}] + \frac{m_f}{2}\mathrm{Tr}[\not{s}]$$

$\downarrow$ zero $\qquad \mathrm{Tr}[\gamma^\mu s_\mu] = 0$

$$= \frac{1}{2} 4 \vec{s}.\vec{p} \qquad \mathrm{Tr}[\not{a}\not{b}] = 4\, a.b$$

$$= 2\, s.p$$

$$= 0 \qquad \text{[since we proved before } s.p = 0\text{]}$$

✓ ✓

3.5) $\mathcal{L} = y\, \bar{\Psi}_1 (1-\gamma_5) \Psi_2 \phi$

In d, dimensions $[L] = [M]^d$

$$[\phi] = [M]^{\frac{d-2}{2}}$$

$$[\bar{\Psi}_1] = [\Psi_2] = [M]^{\frac{d-1}{2}}$$

$$[M]^d \simeq y\, [M]^{\frac{d-2}{2} + d-1}$$

$$\simeq y\, [M]^{\frac{d-2+2d-2}{2}}$$

$$[M]^d \simeq y\, [M]^{\frac{3d-4}{2}}$$

$$[y] = [M]^{d - \frac{(3d-4)}{2}}$$

$$\sim [M]^{\frac{2d-3d+4}{2}}$$

$$\sim [M]^{\frac{4-d}{2}}$$

Now, if we have $d = 4$, then $[y] = 0$ ✓

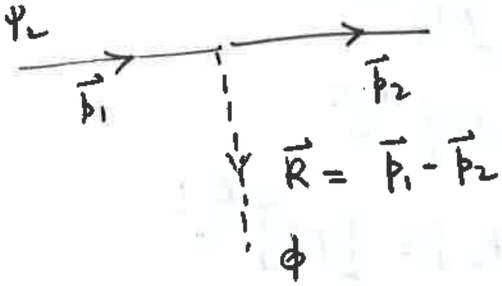
In $\mathcal{D}=4$, y is dimensionless.

3.6) $\psi_1 \longrightarrow \phi\,\psi_2$

This decay is not possible since $\boxed{m_{\psi_2} \gg m_{\psi_1}, m_\phi}$ ✓

3.7) Decay $\psi_2 \longrightarrow \phi\,\psi_1$

Our lowest order Feynman diagram for this decay:



Following Feynman rules on P-288, → srednicki we get

$$i\,\mathcal{T}_{\psi_2 \to \phi + \psi_1} = i\,\bar{u}_{s_2'}(\bar{p}_2')\,y\,(1-\gamma_5)\,u_{s_1}(p_1)\cdot \frac{1}{k^2 - m_\phi^2 - i\epsilon}$$

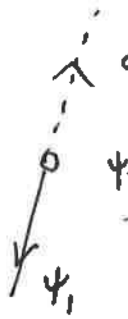
(last factor struck out) ↑ not there!! ✓

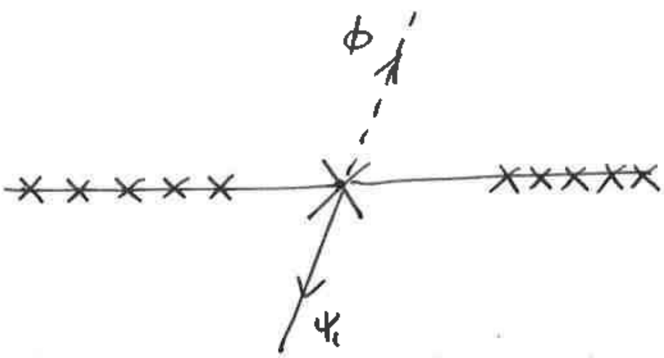
here y = dimensionless ✓.

3.8) Rest frame of ψ_2 (incoming particle).

We can assume $\dfrac{m_1}{m_2}$ and $\dfrac{m_\phi}{m_2}$ to be very small.

Basically, the centre of mass (com) frame will be ψ_2 frame [since other masses are small]





Rest frame of ψ_2.

Clearly $|\bar{p}_i| = |\bar{p}_f|$

$\rightarrow |\vec{p}_\phi| = |\vec{p}_{\psi_1}|$

i.e $\boxed{p_\phi = p'_{2\psi_1}}$

Differential decay rate

$$d\Gamma = \frac{1}{2E1} |\tau|^2 dLIPS_{n'}(k_1)$$

Given $\vec{\lambda} \cdot \vec{p}'_{2\psi_1} = |\vec{\lambda}| \, |\bar{p}'_{2\psi_1}| \cos\theta$

$$\tau = \mathcal{M} = \bar{U}_{s_2}(\bar{p}'_{2\psi_2}) y (1-\gamma_5)$$

$$\tau^* = \bar{\tau} = \bar{V}_{s_1}(p'_i) y (1+\gamma_5)$$

$$\langle|\tau|^2\rangle = \sum_{\substack{\text{outgoing} \\ \text{fermion}}} Tr\left[\bar{U}_{s_2}(\bar{p}'_{2\psi_2}) y (1-\gamma_5) V\bar{s}_1(\bar{p}_i) y (1+\gamma_5)\right]$$

$$d\Gamma = \frac{1}{2E_1} |T|^2 \, dLIPS_2(k)$$

here in ψ_2 frame E_1 = mass of ψ_2 $= m_2$

$dLIPS_2(k) = (2\pi)^4 \, \delta(k_{\psi_p} - k_{\psi_2}) \, d\tilde{k}_1' d\tilde{k}_2'$

$$d\tilde{k} = \frac{d^3k}{(2\pi)^3 \, 2k^0}$$

**

3.9) Intuition: Clearly for polarized case, we expect a dependence on the angle 'θ'. We saw that $\vec{\lambda} \cdot \vec{P}_{\psi_1} = |\vec{\lambda}| \, |\vec{P}_{\psi_1}| \cos\theta$. Hence, we believe that the outcome decay rate will depend on $\underline{\cos^2\theta}$.

Now $\boxed{\cos^2 0 = \cos^2 \pi = 1}$

Hence, at $\theta = 0$ and π, we'll have two things:

1) Decay rate will be maximum OR minimum depending on whether $\cos^2\theta$ is in numerator or denominator

2) Value is same for $\theta = 0$ & $\theta = \pi$

We expect that the measurement is same for θ and $\pi - \theta$, due to azimuthal symmetry; which will be obvious!

→ Think conservation of angular momentum for intuition
amplitude → 0 for configs which can't conserve it.

3.10) Decay rate of unpolarized fermion will not depend on 'θ' because when we sum (average) over all initial possible polarizations we are kind of doing $\langle \cos^2\theta \rangle = \frac{1}{2}$. This eliminates the θ-dependence

✓

Realttempting 3.8

outgoing ψ_1 and ϕ have p' & p''

$$|\mathcal{M}|^2 = \sum_{\substack{\text{outgoing} \\ \text{spin}}} \left| \bar{u}(p') \not{y} (1+\gamma^5) u(p) \right|^2$$

$$= \sum_{\text{spins}} \bar{u}(p') \not{y} (1+\gamma^5) v(p) \bar{v}(p) y (1+\gamma^5) u(p')$$

$$= Tr\left[\not{p}' y (1+\gamma^5) \not{p} y (1+\gamma^5) \right] \longrightarrow$$

$$=$$

This should go down to $\cos^2\theta$ somehow!!

RAGHAV GOVIND JHA

5 May, 2015

1. $\alpha = \frac{g^2}{4\pi}$

$$\frac{dg_i}{d\ln\mu} = \frac{dg}{d\alpha}\frac{d\alpha}{d\ln\mu}$$

$$= -\frac{2\pi}{g}\left(\frac{\alpha^2}{2\pi}\right) Z_i$$

$$\left(\frac{d\alpha}{dg} = \frac{2g}{4\pi} = \frac{g}{2\pi}\right.$$

where $Z_i = \left[\frac{11}{3} C_{adj} - \frac{2}{3}\sum_i C_{f,i} - \frac{1}{6} C_{s,i}\right]_i$

$i = 1.. 2.. 3.$

$$\boxed{\frac{dg_i}{d\ln\mu} = \frac{-g_i^3}{16\pi^2} Z_i} \quad \cdots \quad (1)$$

We need to calculate Z_i for three sectors i.e SU(2), U(1) and SU(3). For brevity, we refer to Z for SU(3) as Z_s, and Z_2 for U(1) and Z_1 for SU(2).

Let's calculate for SU(2) sector

$$Z_1 = \frac{11}{3} C_{adj} - \frac{2}{3}\Sigma C_{f,i} - \frac{1}{6} C_{s,i}$$

$$= \left(\frac{11}{3}\right) 2 - \left(\frac{2}{3}\right)\Sigma C_{f,i} - \frac{1}{6} C_{s,i}$$

For SU(2), we have three fermion (L-handed) doublets (from quark sector) and one LH-fermion doublet from lepton sector giving total of 4.

They're in the fundamental representation of SU(2).

Also, We have one scalar boson (Higgs) doublet).

Hence,

$$Z_1 = \frac{22}{3} - \left(\frac{2}{3}\right)(4)\left(\frac{1}{2}\right) N_g - \frac{1}{6} N_h$$

Setting $N_g = 3$ and $N_h = 1$

$$= \frac{22}{3} - 4 - \frac{1}{6}$$

$$= \frac{44 - 24 - 1}{6}$$

$$= 19/6 \qquad \left(\frac{22}{3} - \frac{4F}{3} - \frac{1}{6}\right)$$

For SU(3):

$$Z_s = \frac{11}{3} C_{adj} - \frac{2}{3} \sum_i C_{f,i} - \frac{1}{6} \underline{\underline{C_{s,i}}}$$

We know that Higgs is a SU(3) singlet or color singlet and hence underlined term above vanishes (not contribute) ; Sum runs over doublets.

$$Z_s = \left(\frac{11}{3}\right) 3 - \frac{2}{3} \sum C_{f,i}$$

$$= \left(\frac{11}{3}\right) 3 - \left(\frac{2}{3}\right) 4 \left(\frac{1}{2}\right) N_g$$

$$= 11 - \frac{4}{3} N_g$$

$$= 7 \qquad (\text{for } N_g = 3)$$

Four triplets of SU(3) for each generation/family.

* Rule: Each quark-lepton family has 4 triplets of SU(3) and 4 doublets of SU(2)

Now, we calculate Z_2 for U(1) sector :-

Note that U(1) abelian has neutral gauge field i.e $\delta A_\mu(x) = 0$. and hence $\frac{11}{3} C_{adj}$ term is 0 here.

$$Z_2 = \frac{-2}{3} \sum C_{f,i} - \frac{1}{6} C_{s,i}$$

Also the U(1) sector to be embedded into SU(5) Model should carry factor of 3/5 and the hypercharge square for each contribution.

$$Z_2 = -\frac{2}{3} \sum_{f,i} \frac{3}{5} Y_f^2 - \frac{1}{6} \sum_{s,i} \frac{3}{5} Y_s^2$$

In standard model, each family has 15 weyl fermions which are representations of $SU(3) \otimes SU(2) \otimes U(1)$. Fermions can be shown to consist of following:

$$(3, 2, +1/6), \quad (1, 2, -1/2), \quad (3, 1, +\tfrac{2}{3}), \quad (3, 1, -\tfrac{1}{3}),$$
$$(1, 1, -1)$$

where $(\cdot , \cdot , \cdot)$ → $SU(3)$, $SU(2)$, $U(1)_y$ → hypercharge

$$Z_2 = \frac{-2}{5}\left[6\left(\frac{1}{6}\right)^2 + 2\left(\frac{-1}{2}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{-1}{3}\right)^2 + (-1)^2\right] N_g$$

$$- \frac{1}{10}\, 2\left(\frac{1}{2}\right)^2 N_h$$

$$= \frac{-2}{5}\left[\frac{1}{6} + \frac{1}{2} + \frac{4}{3} + \frac{1}{3} + 1\right] N_g - \frac{1}{20}$$

$$= -\frac{4}{3} N_g - \frac{1}{20}$$

$N_h = 1$ — Higgs form doublet of complex scalars $(1, 2, \tfrac{1}{2})$

for $N_g = 3$

$$Z_2 = \frac{-81}{20}$$

$$\begin{pmatrix} Z_{U(1)} \\ Z_{SU(2)} \\ Z_{SU(3)} \end{pmatrix} = \begin{pmatrix} Z_2 \\ Z_1 \\ Z_s \end{pmatrix} = \begin{pmatrix} -81/20 \\ 19/6 \\ 7 \end{pmatrix}$$

b) We know from previous assignment that α_i depends on energy scale μ as:

$$\frac{1}{\alpha_i(M_Z)} = \frac{1}{\alpha_i(M_{cutoff})} - \frac{Z_i}{2\pi} \ln\left(\frac{M_{cut}}{M_Z}\right)$$

$\Rightarrow$

$$\frac{1}{\alpha_1(M_Z)} = \frac{1}{\alpha_1(M_{cut})} - \frac{Z_2}{2\pi} \ln Q \qquad \left(Q = \frac{M_{cut}}{M_Z}\right)$$

— (A.1)

and

$$\frac{1}{\alpha_2(M_Z)} = \frac{1}{\alpha_2(M_{cut})} - \frac{Z_1}{2\pi} \ln Q.$$

(A.2)

Subtracting them gives:

$$\frac{1}{\alpha_1(M_Z)} - \frac{1}{\alpha_2(M_Z)} = \frac{Z_1 - Z_2}{2\pi} \ln Q$$

— (C)

At M_{cut} $\alpha_1 = \alpha_2$ or $g_1 = g_2$ which we require

$$* \quad g_2^2 = \frac{e^2}{\sin^2\theta_W}$$

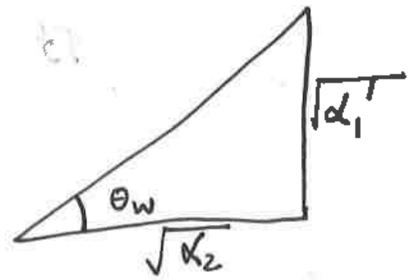
$$\boxed{g^2 = 4\pi\alpha}$$

$$\Rightarrow \quad \alpha_2 = \frac{\alpha_{EM}}{\sin^2\theta_W} \qquad \left(\sin^2\theta_W = 0.231 \right)$$

$$= 0.03357$$

then $\alpha_1' = (0.03357)\tan^2\theta$

$$= 0.0100$$



Using eq. ⓒ and setting the normalization $5/3$, we get,

$$\frac{1}{\alpha_1(M_Z)} - \frac{1}{\alpha_2(M_Z)} = \frac{Z_1 - Z_2}{2\pi} \ln(Q)$$

where $Q = \frac{M_{cutoff}}{M_Z}$.

$$\frac{1}{0.016} - \frac{1}{0.033} = \frac{19/6 + 81/20}{2\pi} \ln Q$$

$$e^{\left(\frac{(29.69)\,2\pi}{3.1666 + 4.05}\right)} = Q$$

$$\Rightarrow \quad Q = e^{\frac{186.45}{7.2166}} \Rightarrow \quad Q \simeq e^{25.8362}$$

But $Q = \dfrac{M_{\text{scale required}}}{91 \text{ GeV}}$

$$\Rightarrow \quad M_{\text{scale}} \Big|_{\substack{\text{at which} \\ g_1 = g_2}} \cong \frac{10^{13} \text{ GeV}}{(1.51 \times 10^{13} \text{ GeV})}.$$

So 10^{13} GeV is the scale at which $\underline{g_1 = g_2}$.

Let's use either of the equations to find $\alpha_1(M_{cut})$ (eq. (A.1) or (A.2))

$$\frac{1}{0.016} = \frac{1}{\alpha_1(M_{cut})} + \frac{(81/20)}{2\pi} \ln Q.$$

$Q \cong e^{25.8362}$

$\ln Q = 25.8362$

$$\Rightarrow \alpha_1(M_{cut}) \approx 0.023$$

So,

$$\frac{1}{\alpha_s(M_Z)} = \frac{1}{0.023} - \frac{7}{2\pi}(25.8362)$$

$$\Rightarrow \quad \boxed{\alpha_s(M_Z) \cong 0.07}$$

Expected

$\cdot\ \alpha_s(M_Z) = 0.11$

The result will be independent of no. of families of fermions i.e N_g since all Z_i's have same contribution of $-\frac{4}{3} N_g$ and we encounter them as differences in our calculation.

2. $V(\Phi) = \frac{1}{2} m^2 Tr(\Phi^2) + \frac{1}{4} \lambda_1 Tr(\Phi^4) + \frac{1}{4} \lambda_2 (Tr\, \Phi^2)^2$ - ①

$$\Phi = v\, diag(\alpha_1 \dots \alpha_5) \quad \dots ②$$

$$\sum_{i=1} \alpha_i = 0 \quad ; \quad \sum_{i=1}^{5} \alpha_i^2 = 1$$

Rewriting eq. ① above using ② we get,

$$V(\Phi) = \frac{1}{2} m^2 v^2 \Sigma \alpha_i^2 + \frac{1}{4} \lambda_1 v^4 \Sigma \alpha_i^4 + \frac{1}{4} v^4 \lambda_2 (\Sigma \alpha_i^2)^2$$

... (2A)

$\frac{dV(\Phi)}{dv} = 0$ gives ;

$$\Rightarrow m^2 v + [\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)] v^3 = 0$$

where $X(\alpha) = \sum_i \alpha_i^4$

$Y(\alpha) = 1$

$$\Rightarrow v^2 = \frac{-m^2}{\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)} \quad \dots ③$$

Using ③ in (2A) we get,

$$V(\Phi) = \frac{1}{2} m^2 v^2 \Sigma \alpha_i^2 + \frac{1}{4} \lambda_1 \frac{m^4 X(\alpha)}{[\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)]^2} + \frac{1}{4} \frac{m^4 \lambda_2 Y(\alpha)}{[\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)]^2}$$

$$= \frac{1}{2} m^2 \left(\frac{-m^2}{\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)} \right) + \downarrow + \downarrow$$

$$= -\frac{1}{2} \frac{m^4}{\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)} + \frac{1}{4} m^4 \frac{X(\alpha)\lambda_1 + \lambda_2 Y(\alpha)}{[X(\alpha)\lambda_1 + Y(\alpha)\lambda_2]^2}$$

$$= -\frac{1}{4} \frac{(m^2)^2}{\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)} \quad \checkmark \ \ldots \ (4)$$

b) To minimize $V(\Phi)$ means that we extremize $\lambda_1 X(\alpha) + \lambda_2 Y(\alpha)$, because then $V(\Phi)$ will get more negative. It means that we just ~~minimize~~ minimize

$$\boxed{\sum_i \alpha_i^4 = X(\alpha)}$$

with constraints (Given also: $\lambda_1, \lambda_2 > 0$)

$$\sum_i \alpha_i^2 = 1 \quad \text{and} \quad \sum_i \alpha_i = 0.$$

Since, we've two constraints, we use Lagrange's multipliers Λ_1 and Λ_2 and extremize

$$\sum_i \frac{1}{4} \alpha_i^4 + \frac{1}{2} \Lambda_1 \sum \alpha_i^2 + \Lambda_2 \alpha_i .$$

After differentiating w.r.t α_i, we get a cubic equation

$$\alpha_i^3 + \Lambda_1 \alpha_i + \Lambda_2 = 0 \quad \text{for each } \alpha_i$$

Also sum of roots of cubic equation $= \frac{-b}{a}$

And since here, we don't have any α_i^2, sum of roots is zero.

Let's base our argument now that only two values of α_i occur. Let's call them A_+ and A_-. Also, let's assume that $A_\pm$ occurs $N_\pm$ times where $N_+ + N_- = 5$.

$$\text{Thus} \quad \sum_i \alpha_i = N_+ A_+ + N_- A_- = 0$$

$$\sum_i \alpha_i^2 = N_+ A_+^2 + N_- A_-^2 = 1$$

$$\Rightarrow \quad A_\pm^2 = \frac{N_\mp}{N_\pm 5} \quad \text{or} \quad A_\pm^4 = \frac{(N_\mp)^2}{25 (N_\pm)^2}$$

Also we can take $\delta > 0$ and $N_+ > N_-$ and write

$$\left.\begin{aligned} N_+ &= \frac{N}{2} + \delta \\ N_- &= \frac{N}{2} - \delta \end{aligned}\right\} N = 5$$

$$\sum_i \alpha_i^4 = N_+ A_+^4 + N_- A_-^4$$

$$= \frac{N_+ (N_-)^2}{25 (N_+)^2} + \frac{N_- (N_+)^2}{25 (N_-)^2}$$

$$= \frac{(N_-)^2}{25 N_+} + \frac{(N_+)^2}{25 N_-}$$

$$= \frac{(N_-)^3 + (N_+)^3}{25 N_+ N_-} = \frac{\left(\frac{N}{2} - \delta\right)^3 + \left(\frac{N}{2} + \delta\right)^3}{25 N_+ N_-}$$

$$= \frac{\frac{N^2}{4} + \frac{3}{2}\delta N^2}{25 N_+ N_-}$$

$$= \frac{\frac{25}{4} + \left(\frac{3}{2}\right) 25\delta}{25\left(\frac{N^2}{4} - \delta^2\right)} = \frac{\frac{25}{4} + \frac{75}{2}\delta}{25\left(\frac{25}{4} - \delta^2\right)}$$

$$= \frac{\frac{1}{4} + \frac{3}{2}\delta}{\frac{25}{4} - \delta^2} = \frac{1 + 6\delta}{25 - 4\delta^2}$$

$\Sigma \alpha_i^4$ increases as <u>δ increases</u>. We need to pick smallest possible positive δ.

$$N_+ = \frac{N}{2} + \delta$$

$$= 2.5 + \delta$$

and $N_- = 2.5 - \delta$

Clearly $\boxed{\delta_{min} = 0.5}$ here and that gives

$\boxed{N_+ = 3}$; $\boxed{N_- = 2}$

We can now follow the same calculation of $\Sigma \alpha_i^4$ for case where we have three possible values of α_i i.e A_+, A_- and A_0.

But the constraint that $\sum_i \alpha_i^4$ should be as ~~less~~ small as possible (minima) will rule out this case of A_+, A_- & A_0. [redacted] Hence, we conclude that given $SU(5)$ breaks as:

$$SU(5) \xrightarrow[\text{minimum}]{\text{at}} SU(3) \otimes SU(2) \otimes U(1)$$

and $\langle \Phi \rangle$ is given by

$$\langle \Phi \rangle = \mathcal{N} v \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \quad \ldots \quad (5)$$

where $\mathcal{N}$ is some extra normalization, we will fix.

Clearly $\sum_i \alpha_i = 0$ for (5)

$$\sum_i \alpha_i^2 = 12 + 18$$
$$= 30 \checkmark.$$

Hence $\mathcal{N} = \sqrt{\frac{1}{30}}$

This gives, $\langle \Phi \rangle = \frac{v}{\sqrt{30}} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$

c) SU(5) has $5^2 - 1 = 24$ generators and

SU(3) ⊗ SU(2) ⊗ U(1) has $8+3+1 = 12$ unbroken generators.

Thus, 12 of the generators of SU(5) become massive.

The gauge boson mass is calculated through the kinetic term

$$\mathcal{L}_{kin} = Tr\left((D_\mu \Phi)(D_\mu \Phi^\dagger)\right)$$

$$= Tr\left(\left(\partial_\mu \Phi + g[A_\mu, \Phi]\right)\left(\partial_\mu \Phi + g[A_\mu, \Phi]\right)\right)$$

$$\simeq g^2 A_\mu^a A^{\mu b}\, Tr\left[[T^a, \langle\Phi\rangle][T^b, \langle\Phi\rangle]\right] \quad \left(A^\mu = A^\mu_a T^a\right)$$

where T^a and T^b are generators (broken ones).

The reason other 12 generators are not important for spectrum of gauge bosons calculation is because, $\langle\phi\rangle$ commutes with those generators and trace is 0 $\Rightarrow$ $\boxed{m=0}$ massless.

Commuting generators correspond to massless gauge field.

In this breaking, 12 massless fields are there.

Let's consider a traceless su(5) generator that does not commute with $\langle\phi\rangle$ and give a mass for the bosons.

$$t = \frac{1}{2}\begin{pmatrix} & & & & -i \\ & & & & \\ & & & & \\ & & & & \\ i & & & & \end{pmatrix}_{5\times5} \in SU(5)$$

and
$$\langle\phi\rangle = \frac{v}{\sqrt{30}}\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

$$[t, \langle\phi\rangle] = \frac{v}{2\sqrt{30}}\begin{pmatrix} & & & & -5i \\ & & & & \\ & & & & \\ & & & & \\ 5i & & & & \end{pmatrix}$$

$$\mathrm{Tr}\left|[t, \langle\phi\rangle]\right|^2 = \frac{v^2}{4(30)}\,50$$

$$= \frac{5v^2}{12}$$

But

$$\mathcal{L}_{\text{mass term}} = \frac{1}{2} m_{\text{boson}}^2 A^2$$

$$\mathcal{L}_{\text{kin}} = g^2 \left(\frac{5}{12} v^2 \right) A^2$$

Comparing we get

$$m^2_{\text{Gauge boson}} = \frac{5 v^2 g^2}{6} \checkmark .$$

Similarly, we can do for other broken generators of SU(5).

3. Four degenerate Dirac fermions in four Euclidean dimensions.

Global symmetry : $SO(4)_{\text{Euclidean}} \otimes SU(4)_{\text{flavor}}$.

Also note that $SO(4)_f$ is a sub-group of $SU(4)_{\text{flavor}}$. **

We can concentrate on $SO(4)_E \otimes SO(4)_{\text{flavor}}$ symmetry to make the selection of diagonal subgroup later more obvious.

Under the aforementioned flavor & Euclidean symmetries, the fermions transform as

$$\Psi_{\alpha i} \longmapsto \xi_{\alpha\beta}\, \Psi^{\beta j}\, \chi^T_{ji}$$

where α, β is the euclidean index, and i, j are flavor indices.

If we do an analogous procedure to twisting and identify $\begin{matrix}\alpha \to i \\ \beta \to j\end{matrix}$, then

$$\boxed{\Psi \to \xi \Psi \xi^T} \quad \cdots\cdots \quad (1)$$

Ψ behaves like a $\underline{4 \times 4}$ matrix and transforms according to (1) above.

** Subgroup of flavor symmetry should coincide with $SO(4)_{\text{Euclidean}}$ **

The subgroup under which fermions can be regarded as 4x4 matrix is clearly:

$$\rightarrow \text{Diagonal}\left(SO(4)_E \otimes SO(4)_{flavor} \right)$$

<u>To construct the action</u> out of these <u>4X4</u> matrices Ψ, γ etc., we'll surely have to trace over them.

$$S = \int d^4x \left\{ \mathrm{Tr}\left(\bar{\Psi} \gamma^\mu \partial_\mu \Psi \right) - m \mathrm{Tr}\left(\bar{\Psi} \Psi \right) \right\} \quad \cdots (2)$$

where Ψ and γ^μ are <u>4X4</u> matrices !!

In the massless limit, (2) becomes

$$S = \int d^4x \ \mathrm{Tr}\left(\bar{\Psi} \gamma^\mu \partial_\mu \Psi \right) \quad \cdots (3)$$

we show that in <u>m=0</u> case, the action in (3) breaks into two independent pieces. we denote,

$$\Psi_+ = P_+ \Psi = \frac{1}{2}\left(\Psi + \gamma_5 \Psi \gamma_5 \right) \quad \cdots (4)$$

$$\Psi_- = P_- \Psi = \frac{1}{2}\left(\Psi - \gamma_5 \Psi \gamma_5 \right) \quad \cdots (5)$$

By abuse of notation, let's denote Ψ_- as Ψ_L and Ψ_+ as** Ψ_R. Then,

$$\bar{\Psi}_L = \Psi_L^\dagger \gamma^0$$
$$= \frac{1}{2}\left(\psi^\dagger - \gamma_5 \psi^\dagger \gamma_5\right)\gamma^0$$
$$= \frac{1}{2}\left(\psi^\dagger \gamma^0 + \gamma_5 \bar{\Psi} \gamma_5\right)$$
$$= \frac{1}{2}\left(\bar{\Psi} + \gamma_5 \bar{\Psi} \gamma_5\right) \quad \cdots \text{⑥}$$

Dirac adjoint
($\bar{\Psi} = \psi^\dagger \gamma^0$)
[Note that $(\gamma^5)^\dagger = \gamma^5$]
$\{\gamma^5, \gamma^0\} = 0$

and similarly,

$$\bar{\Psi}_R = \frac{1}{2}\left(\Psi - \gamma_5 \Psi \gamma_5\right) \quad \cdots \text{⑦}$$

If the action in ③ has to break into two independent pieces, we can have following combinations.

$$\text{Tr}\left(\bar{\Psi}_+ \gamma_\mu \partial_\mu \Psi_-\right), \quad \text{Tr}\left(\Psi_- \gamma_\mu \partial_\mu \Psi_+\right)$$

or

$$\text{Tr}\left(\bar{\Psi}_\pm \gamma_\mu \partial_\mu \Psi_\pm\right)$$

We go back to old Ψ_+ and Ψ_-
$\Psi_+ \longmapsto \Psi_R$
$\Psi_- \longmapsto \Psi_L$

We'll show now that underlined terms only become part of Eq. ③

** To not confuse "†" (dagger) with "+" etc. **

A better argument can be also given on the basis of fact that $P_-^2 = P_-$; but $P_+ P_- = P_- P_+ = 0$.

We'll try to write out $Tr(\bar{\Psi}_+ \gamma_\mu \partial_\mu \Psi_-)$ explicitly.

$$Tr(\bar{\Psi}_+ \gamma_\mu \partial_\mu \Psi_-) = \frac{1}{2}(\bar{\Psi} - \gamma_5 \Psi \gamma_5) \gamma_\mu \partial_\mu \frac{1}{2}(\Psi + \gamma_5 \Psi \gamma_5)$$

$$= \frac{1}{2} Tr(\bar{\Psi} \gamma_\mu \partial_\mu \bar{\Psi})$$

$$= \frac{1}{2} Tr(\bar{\Psi} \gamma^\mu \partial_\mu \bar{\Psi}) \quad \ldots (8)$$

$\{ Tr(\gamma^5(\text{odd no. of } \gamma^\mu)) = 0, \ (\gamma^5)^2 = \mathbb{1} \}$

Similarly,

$$Tr(\bar{\Psi}_- \gamma^\mu \partial_\mu \Psi_+) = \frac{1}{2} Tr(\Psi \gamma^\mu \partial_\mu \Psi) \quad \ldots (9)$$

Using (8) & (9) in (3), we get

$$S = \int d^4x \left\{ Tr(\bar{\Psi}_+ \gamma^\mu \partial_\mu \Psi_-) + Tr(\bar{\Psi}_- \gamma^\mu \partial_\mu \Psi_+) \right\}$$

✓

RAGHAV

(10)

1) Show that the advanced propagator defined by

$$D_{adv}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2 - m^2 - i\,\mathrm{sgn}(k_0)\varepsilon} \quad - (1)$$

non-zero only for $x_0 > 0$

$\mathrm{Sgn}(x) = +1 \quad x > 0$
$\quad = -1 \quad x < 0$

$\mathrm{Sgn}(k_0) = +1$ if $k_0 > 0$
$\quad = -1 \quad k_0 < 0$

Sol^n: Looking at the poles of (1)

$$k^2 - m^2 - i\,\mathrm{sgn}(k_0) = 0$$

$$k_0^2 - (\vec{k}^2 + m^2) - i\,\mathrm{sgn}(k_0)\varepsilon = 0$$

$$\Rightarrow k_0^2 = \underbrace{\vec{k}^2 + m^2}_{\omega_k^2} + i\,\mathrm{sgn}(k_0)\varepsilon$$

$$= \omega_k^2 + i\,\mathrm{sgn}(k_0)\varepsilon$$

$$k_0 = \pm\sqrt{\omega_k^2 + i\,\mathrm{sgn}(k_0)\varepsilon}$$

$$= \pm\sqrt{\omega_k^2 \pm i\varepsilon}$$

$$\simeq \pm\left(\omega_k \pm \frac{i\varepsilon}{2\omega_k}\right)$$

$$\simeq \pm\omega_k + i\varepsilon' \quad \left(\varepsilon' = \frac{\varepsilon}{2\omega_k}\right) ;\ x_0 > 0$$

$\varepsilon' > 0$

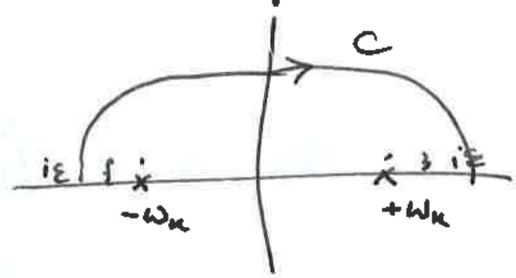
The two poles are in the upper half of the complex plane hitting the axes when $\epsilon \to 0$.

So,

$$D_{adv}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik^0 x_0}}{k^2 - m^2 - i\,\mathrm{sgn}(k_0)\,\epsilon}$$

For this integral to converge, the exponential dependence should not dominate $1/k^2$. Hence positive values of x_0 ($x_0 > 0$) mean that we need positive imaginary component of k^0.

But, if that is true, then the integral will be evaluated when the contour wraps the two poles above the plane.



This will give a non-zero result. Note that for $x_0 < 0$, contour would have been closed in the lower plane giving zero !!

2) $$Z(J) = \int \mathcal{D}\phi \, e^{-S + \int J(x)\varphi(x) d^d x}$$

$$\Rightarrow \quad \frac{1}{Z[0]} \frac{\delta Z[J]}{\delta J(x_1)}\bigg|_{J=0} = \underbrace{\frac{1}{Z[0]} \int \mathcal{D}\varphi \, \varphi(x_1) e^{-S[\phi]}}_{\to \langle \phi(x_1) \rangle}$$

Similarly.

$$\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \frac{1}{Z[0]} \int \mathcal{D}\varphi \, \varphi(x_1)\varphi(x_2) e^{-S[\phi]} \quad - ①$$

$$= \langle \phi(x_1)\varphi(x_2) \rangle$$

we can write using the concept of time-ordering as

$$\frac{1}{Z[0)} \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \langle \varphi(x_1)\varphi(x_2) \rangle = \langle 0 | T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)] | 0 \rangle \quad - ②$$

Eq. ② basically means that functional derivatives of $Z[J]$ give vacuum expectation values of time-ordered product of field operators.

Note:

<u>Euclidean</u> <u>Propagator</u> $\left(e^{-S/\hbar} \overset{\hbar=1}{\approx} e^{-S[\phi]} \right)$

$$Z_0[J] = \int \mathcal{D}\varphi \, e^{-\int \left[\frac{1}{2}(\partial\varphi)^2 + \frac{m^2\varphi^2}{2}\right] d^d x + \int J(x)\varphi(x) d^d x}$$

$$\tilde{\varphi}(p) = \int d^d x \; e^{-ip.x} \; \varphi(x)$$

$$\varphi(x) = \int \frac{d^d p}{(2\pi)^d} \; e^{ip.x} \; \tilde{\varphi}(p)$$

Term in the exponential is then:

$$-\int \frac{d^d p}{(2\pi)^d} \left[\frac{1}{2} \tilde{\phi}(p)(p^2+m^2)\tilde{\varphi}(-p) - \tilde{J}(p)\tilde{\varphi}(-p) \right]$$

Complete the square of the term in square brackets:

$$\frac{1}{2}\left[\tilde{\varphi}(p) - \frac{1}{p^2+m^2}\tilde{J}(p)\right](p^2+m^2)\left[\tilde{\varphi}(-p) - \frac{1}{p^2+m^2}\tilde{J}(-p)\right]$$

$$- \frac{1}{2}\tilde{J}(p)\frac{1}{p^2+m^2}\tilde{J}(-p)$$

↳ subtracted this to fix everything.

Now let

$$\tilde{\varphi}(p) = \tilde{\phi}(p)' + (p^2+m^2)^{-1}\tilde{J}(p)$$

$$Z_0[J] = \int \mathcal{D}\tilde{\varphi}' \; e^{-\frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p)'(p^2+m^2)\tilde{\phi}(-p)' + \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

$$= Z_0[0] \; e^{\frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

↓ x-space

$$Z_0[J] = Z_0[0] \; e^{\frac{1}{2}\int d^d x' \int d^d x'' J(x')\Delta(x'-x'')J(x'')}$$ gives $\Delta(x'-x'') = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot(x'-x'')}}{p^2+m^2}$.

So, the Euclidean propagator is :

$$D_E(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik.(x-y)}}{K^2+m^2}$$

→ no '$i\varepsilon$' prescription required.

Space average of $D_E(x-y)$ means, roughly ;

$$\int_{\text{'space average'}} D_E(x-y) = \underline{\underline{D_E(\tau-\tau')}}$$

(four vector → $x-y$; Euclidean time → τ)

↓ required.

$$\bar{D}_E(\tau\tau') = \int \frac{dk^0}{(2\pi)} \frac{e^{-ik^0(\tau-\tau')}}{\omega^2+m^2} \quad ; k^0 := \omega$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(\tau'-\tau)}}{\omega^2+m^2}$$

So the integral can be evaluated for $\tau'>\tau$ by closing in the upper half plane & $\tau'<\tau$ by closing in the lower half plane

Residues at $\omega = \pm im$

Digression :-

$$f(x,t) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{f}(k,t)$$

Space average is :

$$\bar{f}(t) = \int d^3x \; f(\vec{x},t)$$

$$= \int d^3x \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{f}(k,t)$$

$$= \frac{1}{(2\pi)^3} (2\pi)^3 \delta^3(\vec{k}) \tilde{f}(k,t)$$

$$= \tilde{f}(0,t)$$

Space average of $D_E(x-y)$ just amounts to doing the dk^0 integration when space-components are set to zero.

$$\bar{D}_E(\tau-\tau') = \underbrace{\theta(\tau-\tau')\frac{e^{\omega(\tau'-\tau)}}{2\omega}}_{\text{lower half}} + \underbrace{\theta(\tau'-\tau)\frac{e^{\omega(\tau-\tau')}}{2\omega}}_{\text{upper half}}$$

$$= \frac{e^{-\omega|\tau-\tau'|}}{2\omega}$$

(No 'i' dependence on exponent since $\omega = \pm im$ takes that out).

$$\bar{D}_E(\tau-\tau') = \frac{e^{-\omega|\tau-\tau'|}}{2\omega}$$

✓

→ function of Euclidean time.

Now, if we substitute $\tau = it$ and $\tau' = it'$

we get,

$$D_{\text{Minkowski}}(t-t') = \frac{e^{-i\omega|t-t'|}}{2\omega}$$

✓

3. $Z = e^{\frac{1}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)}$

and $E = -\frac{1}{\beta} \ln Z$

$$= \frac{-1}{2\beta} \int d^4x\, d^4y\; J(x)\, D(x-y)\, J(y) \qquad - \text{①}$$

Now, it makes more sense to pass over to k-space and solve this. Writing $D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\varepsilon}$

and using in ①, we get:

$$E = \frac{-1}{2\beta} \underbrace{\int d^4x\, J(x)\, e^{ikx}} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-iky}}{k^2 - m^2 + i\varepsilon} \int d^4y\, J(y)$$

But note that ⁓ is the fourier transform of $J(x)$ i.e $\tilde{J}(k)$ and similarly for $J(y)$.

We then have;

$$E = \frac{-1}{2\beta} \int d^4k \, \frac{\tilde{J}(k)\, \tilde{J}(-k)}{k^2 - m^2 + i\varepsilon}$$

$$= \frac{-1}{2\beta} \int d^4k \, \frac{|\tilde{J}(k)|^2}{k^2 - m^2 + i\varepsilon} \qquad - \text{②}$$

Now, we use the $J(x)$ given in the problem.

$$J(x) = \theta(T-t)\,\theta(T+t)\left\{a_1\,\delta(x) + a_2\,\delta(x-R)\right\}$$

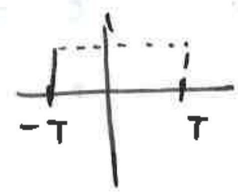
Let's take fourier transform of $J(x)$

$$\tilde{J}(k) = \int \frac{d^4x}{(2\pi)^4}\, J(x)\, e^{ik\cdot x}$$

$$= \int_{-T}^{T} \frac{dt\, e^{ik^0x_0}}{(2\pi)} \int \frac{d^3x}{(2\pi)^3} \left\{a_1\,\delta(x) + a_2\,\delta(x-R)\right\} e^{-i\vec{k}\cdot x}$$

$\theta(T-t) = 1$ when $T - t > 0$, $\boxed{T > t}$

$\theta(T+t) = 1$ if $T + t > 0$, $T > -t$, $\boxed{-T < t}$



Let us define $k^0 := \omega$

$$= \left| \frac{e^{i\omega t}}{i\omega} \right|_{-T}^{T} \int \frac{d^3x}{(2\pi)^3} \left\{a_1\delta(x) + a_2\delta(x-R)\right\} e^{-i\vec{k}\cdot x}$$

$$= \frac{2\sin\omega T}{2\omega\pi}\left[\int \frac{d^3x}{(2\pi)^3}\, a_1\delta(x)\, e^{-i\vec{k}\cdot x} + \int \frac{d^3x}{(2\pi)^3}\, a_2\,\delta(x_{-R})\, e^{-i\vec{k}\cdot x}\right]$$

$$= \frac{\sin\omega T}{\omega\pi}\left[a_1 + a_2\, e^{-i\vec{k}\cdot\vec{R}}\right]$$

$$\tilde{J}(k)\,\tilde{J}(-k) = |\tilde{J}(k)|^2 = \frac{\sin^2\omega T}{\omega^2\pi^2}\left[a_1^2 + a_2^2 + 2a_1a_2\cos\vec{k}\cdot\vec{R}\right]$$

Plugging ③ to ② we get,

$$E = \frac{-1}{2\beta} \int d^4k \; \frac{\sin^2 \omega T}{\omega^2 \pi^2} \left[a_1^2 + a_2^2 + 2a_1a_2 \cos \bar{k}\cdot\bar{R}\right] \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$= \frac{-1}{2\beta} \int d\omega \; \frac{\sin^2 \omega T}{\omega^2 \pi^2} \int d^3k \left[\frac{a_1^2 + a_2^2 + 2a_1a_2 \cos \vec{k}\cdot\vec{R}}{k^2 - m^2 + i\varepsilon}\right] \quad \cdots ④$$

ω^2 still here!

But we know

$$\int d\omega \; \frac{\sin^2(\omega T)}{\omega^2} = \pi T \quad \cdots ⑤$$

[standard integral]

using ⑤ in ④ we get,

$$E = \frac{-1}{2\beta} \frac{T}{\pi} \int d^3k \; \frac{a_1^2 + a_2^2 + 2a_1a_2 \cos \vec{k}\cdot\vec{R}}{k^2 - m^2 + i\varepsilon}$$

$$= \underbrace{\frac{-1}{2\beta} \frac{T}{\pi} \int d^3k \; \frac{a_1^2 + a_2^2}{k^2 - m^2 + i\varepsilon}}_{I_1} \; \underbrace{- \frac{2a_1a_2}{2\beta} \frac{T}{\pi} \int \frac{d^3k \cos \vec{k}\cdot\vec{R}}{k^2 - m^2 + i\varepsilon}}_{I_2}$$

We can do I_1 but it does not make any sense since it has no R dependence and it will anyways fall out in calculation of force..

$$F = \frac{-\partial E}{\partial R} \cong \frac{-\partial}{\partial R}\left[\frac{-a_1 a_2 T}{2\beta\pi}\int d^3k \;\frac{e^{i\vec{k}\cdot\vec{R}} + e^{-i\vec{k}\cdot\vec{R}}}{k^2 - m^2 + i\varepsilon}\right]$$

Using Integral table, we evaluate and find in the lt $\epsilon \to 0$.

Note $T >> \frac{1}{m}$ $\Rightarrow m >> \omega$ $(k^0)^2 << (m)^2$.

$$\cong \frac{-\partial}{\partial R}\left[\frac{-a_1 a_2 T}{2\beta\pi}\;\frac{e^{-m|R|}}{2\pi R}\right]$$

$$\cong \frac{a_1 a_2 T}{2\pi\beta}\left[\frac{-m e^{-m|R|}}{2\pi R} - \frac{e^{-m|R|}}{4\pi R^2}\right]$$

$$\cong \frac{-a_1 a_2 T}{2\pi\beta}\left[\frac{m e^{-m|R|}}{2\pi R} + \frac{e^{-m|R|}}{4\pi R^2}\right]$$

$\cong -$ (⊕ve quantity !!)

The force dies out with increasing $|\bar{R}|$ ..

$$R >> \frac{1}{m} \Rightarrow m\;R >> 1$$

$$-m\,|R| <<< 1$$

↳ dies out sufficiently quickly..

Attractive since it has a negative sign in front !!

Note :- Later realized that the form given for 'Z' does not have a factor of "i". This clearly hints that we should work in the Euclidean space where propagator has factor $\frac{1}{k^2+m^2}$..

I did it using Minkowski propagator & wick rotated back later.

✓

10th Feb 2015

(1) $\phi_n^c = \frac{\delta W(J)}{\delta J_n} = \langle \Phi \rangle_J$

if $\Gamma(\varphi_c) = W(J) - \sum_n J_n \phi_n^c$

a) Show that

$$\frac{\delta \Gamma}{\delta \phi_m^c} = -J_m$$

Sol^n :-

$$\frac{\delta \Gamma}{\delta \phi_m^c} = \frac{\delta W(J)}{\delta \phi_m^c} - \sum_n \frac{\delta}{\delta \phi_m^c}\left(J_n \phi_n^c\right)$$

$$= \frac{\delta W(J)}{\delta J_n}\frac{\delta J_n}{\delta \phi_m^c} - \sum_n J_n \frac{\delta \phi_n^c}{\delta \phi_m}$$

$$= \phi_n^c \frac{\delta J_n}{\delta \phi_m^c} \rightarrow 0 - \sum_n J_n \delta_{nm}$$

$$= -J_m \checkmark$$

b) Verify

$$\sum_n \frac{\delta^2 W(J)}{\delta J_1 \delta J_n} \frac{\delta^2 \Gamma}{\delta \phi_n^c \delta \phi_2^c} = -\delta_{12}$$

Sol^n: $\sum_n \frac{\delta}{\delta J_1} \frac{\delta W(J)}{\delta J_n} \quad \frac{\delta}{\delta \phi_2^c} \frac{\delta \Gamma}{\delta \phi_n^c}$

$$\sum_n \frac{\delta}{\delta J_1} \phi_n^c \ \frac{\delta}{\delta \phi_2^c}(-J_n)$$

$$-\sum_n \frac{\delta \phi_n^c}{\delta \phi_2^c} \frac{\delta J_n}{\delta J_1}$$

$$-\sum_n \delta_{2,n} \delta_{n,1}$$

$$-\delta_{12} \quad ✓$$

c) Using the result

$$\frac{\partial}{\partial \alpha} M^{-1}(\alpha) = -M^{-1} \frac{\partial M}{\partial \alpha} M^{-1} \quad \cdots \text{(A)}$$

where $M_{\alpha\beta} = \frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta \phi_\beta}$ and $M^{-1}_{\alpha\beta} = \frac{-\delta^2 W}{\delta J_\alpha \delta J_\beta}$

Sol^n: Using the eq^n (A) given;

$$\frac{\partial M^{-1}_{12}}{\partial J_3} = -M^{-1}_{1\alpha} \frac{\partial M_{\alpha\beta}}{\partial J_3} M^{-1}_{2\beta}$$

$$\frac{\partial}{\partial J_3} M_{12}^{-1} = \sum_{\alpha,\beta,\xi} - M_{1\alpha}^{-1} \frac{\partial M_{\alpha\beta}}{\partial \phi_\xi} \frac{\partial \phi_\xi}{\partial J_3} M_{2\beta}^{-1} \quad \cdots \quad (B)$$

Now $M_{12}^{-1} = \dfrac{-\delta^2 W}{\delta J_1 \delta J_2}$

$$M_{1\alpha}^{-1} = \frac{-\delta^2 W}{\delta J_1 \delta J_\alpha}$$

$$M_{2\beta}^{-1} = \frac{-\delta^2 W}{\delta J_2 \delta J_\beta}$$

Using above three eqⁿs in (B) we get,

$$\frac{-\partial}{\partial J_3}\left(\frac{\delta^2 W}{\delta J_1 \delta J_2}\right) = \sum_{\alpha,\beta,\xi} - \frac{\delta^2 W}{\delta J_1 \delta J_\alpha} \underline{\underline{\frac{\delta M_{\alpha\beta}}{\delta \phi_\xi}}} \, \underline{\underline{\frac{\delta \phi_\xi}{\delta J_3}}} \frac{\delta^2 W}{\delta J_2 \delta J_\beta} \quad - \quad (C)$$

Let's calculate underlined terms

$$\frac{\delta M_{\alpha\beta}}{\delta \phi_\xi} = \frac{\delta}{\delta \phi_\xi} \frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta \phi_\beta}$$

$$= \Gamma_{\alpha\beta\xi} \quad - \quad (D)$$

and

$$\frac{\delta \phi_\xi}{\delta J_3} = \frac{\delta}{\delta J_3} \frac{\delta W}{\delta J_\xi}$$

$$= \frac{\delta^2 W}{\delta J_3 \delta J_\xi} = -G_{\xi 3} \quad - \quad (E)$$

Using (D) & (E) in (C) we get,

$$-\frac{\delta^2 W(J)}{\delta J_1 \delta J_2 \delta J_3} = \sum_{\alpha,\beta,\xi} +\frac{\delta^2 W}{\delta J_1 \delta J_\alpha} \Gamma_{\alpha\beta\xi} G^c_{\xi 3} \frac{\delta^2 W}{\delta J_\beta \delta J_2}$$

Change $\xi \to \gamma$, dummy.

$$\Rightarrow G^3_C(1,2,3) = \sum_{\alpha,\beta,\gamma} G_c^{(2)}(1,\alpha) G_C^{(2)}(2,\beta) G_c^{(2)}(3,\gamma) \Gamma^{(3)}_{\alpha\beta\gamma}$$

✓

2) Dirac Equation

$$(i\not{\partial} + m)\psi = 0$$

In momentum space $(\not{p} + m)\psi(p) = 0$

Rest frame of particle $p = (m, 0, 0, 0)$; since it is Lorentz invariant

$$\begin{aligned} \not{p} &= \gamma^{\mu} p_{\mu} \\ &= \gamma^{0} p_{0} + \cdots \\ &= \gamma^{0} m \end{aligned}$$

This means ; $(\gamma^{0} + \mathbb{I})\psi(p) = 0$

using $\gamma^{0} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$; we get

$$\begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix} \psi(p) = 0$$ (excluding factors of 2 in coefficient)

↓

This clearly has been reduced from 4x4 to 2x2 non-zero matrix ($\mathbb{I}_2$) ; hence 2 d.o.f.

Two of the four components of ψ are zero.

Also, note that $(\gamma^0 + \mathbb{I})^2$ is upto some factor equal to $(\gamma^0 + \mathbb{I})$.

i.e $\quad A^2 = A \quad$; Projection operator

Hence, it projects original 4 component spinor to a subspace where it is only 2 component hence in agreement with Wigner.

2) <u>Rarita Schwinger</u> Spin 3/2

$$\mathcal{L} = \bar{\Psi}_\mu \gamma^{\mu\nu\lambda} \partial_\nu \Psi_\lambda + m \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu$$

E.O.M is given by (E-L w.r.t $\bar{\Psi}$) Euler-Lagrange equation

$$\gamma^{\mu\nu\lambda} \partial_\nu \Psi_\lambda + m \gamma^{\mu\nu} \Psi_\nu = 0$$

(Relabelling of indices)..

$$\Rightarrow \quad \gamma^{\mu\lambda\nu} \partial_\lambda \Psi_\nu - m \gamma^{\mu\nu} \Psi_\nu = 0$$

$$\Rightarrow \left(\gamma^{\mu\lambda\nu} \partial_\lambda - m \gamma^{\mu\nu} \right) \Psi_\nu = 0$$

✓.

b) Act on E.O.M with ∂_μ

$$\partial_\mu \left(\gamma^{\mu\lambda\nu} \partial_\lambda - m \gamma^{\mu\nu} \right) \Psi_\nu = 0$$

↓ Goes to zero ($\partial_\mu \partial_\lambda$ is symmetric under change of indices and $\gamma^{\mu\lambda\nu}$ is antisymmetric)

$$\Rightarrow \quad \partial_\mu \gamma^{\mu\nu} \Psi_\nu = 0 \quad \ldots \quad (1)$$

Write $\gamma^{\mu\nu} = \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} - \gamma^\nu \gamma^\mu$

$$= \frac{1}{2} 2 \eta^{\mu\nu} \mathbb{I} - \gamma^\nu \gamma^\mu$$

$$= \eta^{\mu\nu} - \gamma^\nu \gamma^\mu \quad \cdots \quad ②$$

Now using ②, ① becomes

$$\partial_\mu \left(\eta^{\mu\nu} \psi_\nu - \gamma^\mu \gamma^\nu \psi_\nu \right) = 0$$

$$\Rightarrow \quad \partial_\mu \psi^\mu - (\gamma^\mu \partial_\mu)(\gamma^\nu \psi_\nu) = 0$$

$$\Rightarrow \quad \partial_\mu \psi^\mu = (\gamma^\mu \partial_\mu)(\gamma^\nu \psi_\nu) \checkmark .$$

c) Act on E.o.M with γ_μ.

Few identities to be used:

* $\gamma^{\mu\nu\lambda} = \gamma^{[\mu} \gamma^\nu \gamma^{\lambda]} = \frac{1}{3} \left(\gamma^\mu \gamma^{\nu\lambda} + \gamma^\nu \gamma^{\lambda\mu} + \gamma^\lambda \gamma^{\mu\nu} \right)$

→ cyclic permutation of indices.. μ, ν, λ

* $\gamma^\lambda \gamma^\nu + \gamma^\nu \gamma^\lambda = 2\eta^{\nu\lambda}$
* $\gamma^\mu \gamma_\mu = 4 \mathbb{I}_4$
* $\gamma^\mu \gamma^\nu \gamma_\mu = -2 \gamma^\nu$

E.O.M reads :

$$(\gamma^{\mu\lambda\nu}\partial_\lambda - m\gamma^{\mu\nu})\psi_\nu = 0$$

Act with γ_μ

$$\gamma_\mu(\gamma^{\mu\lambda\nu}\partial_\lambda - m\gamma^{\mu\nu})\psi_\nu = 0$$

$$\gamma_\mu\left\{\frac{1}{3}(\gamma^\mu\gamma^{\lambda\nu} + \gamma^\lambda\gamma^{\nu\mu} + \gamma^\nu\gamma^{\mu\lambda})\partial_\lambda\psi_\nu\right\} - m\gamma_\mu\gamma^{\mu\nu}\psi_\nu = 0$$

From here onward, factors are not taken care of since we have to just prove $\boxed{\gamma^\mu\partial_\mu = 0}$

$$(\gamma_\mu\gamma^\mu\gamma^{\lambda\nu} + \gamma_\mu\gamma^\lambda\gamma^{\nu\mu} + \gamma_\mu\gamma^\nu\gamma^{\mu\lambda})\partial_\lambda\psi_\nu - m\gamma_\mu\left(\frac{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu}{2}\right)\psi_\nu = 0$$

$$\left[4\gamma^{\lambda\nu} + \gamma_\mu\gamma^\lambda\left(\frac{1}{2}\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) + \gamma_\mu\gamma^\nu\left(\frac{1}{2}\gamma^\mu\gamma^\lambda - \gamma^\lambda\gamma^\mu\right)\right]\partial_\lambda\psi_\nu - \frac{m\gamma_\mu}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\psi_\nu = 0$$

$$- \frac{m}{2}(4\gamma^\nu + 2\gamma^\nu)\psi_\nu = 0$$

Simplifying

$$\left[4\gamma^{\lambda\nu} + \gamma^\nu\gamma^\lambda + 2\eta^{\nu\lambda} - \gamma^\nu\gamma^\lambda - 2\eta^{\nu\lambda}\right]\partial_\lambda\psi_\nu - 3m\gamma^\nu\psi_\nu = 0$$

Dropping all factors for convenience,

$$\gamma^{\lambda\nu}\partial_\lambda\psi_\nu = \gamma^\nu\psi_\nu$$

Now the L.H.S = 0 ; from Eq. (1) on Page -(7)

$$\Rightarrow \quad \gamma^{\mu} \psi_{\mu} = 0 \qquad (\text{relabell } \mu \mapsto \nu \text{ since dummy.})$$

d) Constraint 1 : $\partial_{\mu} \psi^{\mu} = (\gamma^{\mu} \partial_{\mu})(\gamma^{\nu} \psi_{\nu})$ .... (1)

Constraint 2 : $\gamma^{\mu} \psi_{\mu} = 0$ ... (2)

The $\gamma^{\mu} \psi_{\mu} = 0$ constraint is like a gauge; infact harmonic gauge. Interesting to note is that (2) in (1) implies $\boxed{\partial_{\mu} \psi^{\mu} = 0}$

So, now we have

$$\boxed{\partial_{\mu} \psi^{\mu} = 0} \quad \text{and} \quad \boxed{\gamma^{\mu} \psi_{\mu} = 0}$$

Note : we have suppressed the spinor index 'α' on ψ_{μ} all throughout.

Initially ψ_{μ}^{α} had 16 d.o.f.

$\downarrow \partial_{\mu} \psi^{\mu} = 0$

reduced it to 12

and then the second constraint reduced it further to 8 $(16 \to 12 \to 8)$ ..

But, we need to do more..

e) Including the constraints in the E.O.M derived in part (a) gives :

$$-\gamma^{\mu\lambda\nu}\partial_\lambda \psi_\nu + m\gamma^{\mu\nu}\psi_\nu = 0 \qquad \leftarrow \text{E.O.M}$$

Impose $\partial . \psi = 0$ i.e $\partial_\mu \psi^\mu = 0$

and $\gamma^\mu \psi_\mu = 0$

$$m\gamma^{\mu\nu}\psi_\nu \longrightarrow \frac{m}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\psi_\nu$$

$$= \frac{m}{2}\left(2\gamma^\mu\gamma^\nu - 2\eta^{\mu\nu}\right)\psi_\nu$$

$$= m\left(\gamma^\mu\gamma^\nu\psi_\nu - \psi^\mu\right)$$

$$= m\left(\gamma^\mu(\gamma . \psi) - \psi^\mu\right)$$

But $\gamma . \psi = 0$

So $m\gamma^{\mu\nu}\psi_\nu \longmapsto -m\psi^\mu$ ✓.

The $-\gamma^{\mu\lambda\nu}\partial_\lambda \psi_\nu$ part gives (leaving out factors like $\frac{1}{3}$ etc.

$$-\left[\gamma^\mu\gamma^{\lambda\nu} + \gamma^\lambda\gamma^{\nu\mu} + \gamma^\nu\gamma^{\mu\lambda}\right]\partial_\lambda\psi_\nu$$

$$-\left\{\gamma^\mu(\gamma^\lambda\gamma^\nu - \eta^{\lambda\nu}) + \gamma^\lambda(\gamma^\nu\gamma^\mu - \eta^{\nu\mu}) + \gamma^\nu(\gamma^\mu\gamma^\lambda - \eta^{\mu\lambda})\right\}\partial_\lambda\psi_\nu$$

$$-\Big[\gamma^\mu\,\cancel{(\gamma\cdot\partial)(\gamma\cdot\psi)} - \cancel{\partial^\nu\psi_\nu} + \gamma^\lambda\,\cancel{(\gamma\cdot\psi)(\gamma\cdot\partial)}$$
$$- \gamma^\lambda\partial_\lambda\psi^\mu$$
$$+ \gamma^\nu\,\cancel{(\gamma\cdot\psi)(\gamma\cdot\partial)} - \cancel{\gamma^\nu\partial^\mu\psi_\nu}$$

we get

$$-\gamma^{\mu\lambda\nu}\partial_\lambda\psi_\nu \longmapsto +\gamma^\lambda\partial_\lambda\psi^\mu$$
$$+\not{\partial}\psi^\mu$$

So, constrained E.O.M is

$$(\not{\partial} - m)\psi^\mu = 0$$

↳ Note that spinor index on ψ^μ is still suppressed.

We can remove the last four d.o.f following the method we did for Dirac eq^n.

Taking Fourier transform of $(\not{\partial} - m)\psi^\mu = 0$

gives $(i\not{p} - m)\psi^\mu = 0$

going to a frame where $p^\mu = (m, 0, 0, 0)$

gives

$$(i\gamma^0 - \mathbb{I})\psi^\mu = 0 \quad *$$ (there should not be an 'i' here)..

! Following same prescription as discussed in Dirac case we can show that

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$A = i\gamma^0 - \mathbb{I} = A^2$$

(Projection operator)

and it reduces the d.o.f from 8 to 4..

Note :- The factor of 'i' survives since $\mathcal{L}$ had no 'i' but, don't know the interpretation..

We remain with $8 - 4 = 4$ d.o.f which is the correct for spin $3/2$ i.e $2(\frac{3}{2}) + 1 = 4$ ✓.

(20)

RAGHAV..

1) $$\frac{d\alpha}{d\ln\mu} = -2\alpha\left(\left(\frac{11C_A - 2N_f}{3}\right)\frac{\alpha}{4\pi} + b_2\left(\frac{\alpha}{4\pi}\right)^2 + \cdots\right) \quad - (1)$$

where $\alpha = \frac{g^2}{(4\pi)^2}$

$C_A = 3$, $C_f = \frac{4}{3}$ } for SU(3)

Expression for $\alpha(\mu)$ ...

Writing (1) for the one-loop ;

$$\mu\frac{d\alpha}{d\mu} = -2\alpha\left(\left(\frac{11C_A - 2N_f}{3}\right)\frac{\alpha}{4\pi}\right)$$

$$\mu\frac{d\alpha}{d\mu} = \frac{-\alpha^2}{2\pi}\left(\left(\frac{11C_A - 2N_f}{3}\right)\right)$$

(the bracket $\rightarrow \chi$)

$$\Rightarrow \int_{\alpha_0}^{\alpha} \frac{-d\alpha}{\alpha^2\chi} = \frac{1}{2\pi}\int_{\mu_0}^{\mu}\frac{d\mu}{\mu} \quad ✓$$

$$\frac{1}{\chi}\left|\frac{1}{\alpha}\right|_{\alpha_0}^{\alpha} = \frac{1}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{\chi}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right) = \frac{1}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{\alpha} = \frac{1}{\alpha_0} + \frac{\chi}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right) \quad ✓$$

~~$\alpha_0\alpha$~~ $\Rightarrow \alpha_0 = \alpha + \frac{\chi\alpha\alpha_0}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)$

$$\Rightarrow \alpha(\mu) = \frac{\alpha_0}{1 + \alpha_0\chi\ln(\mu/\mu_0)}$$

10) for $N = 2$

$C_A = 2$

$C_f = 3/4$

β- function develops a zero when the signs of b_0 & b_1 are opposite. In this regime, the asymptotic freedom is lost.

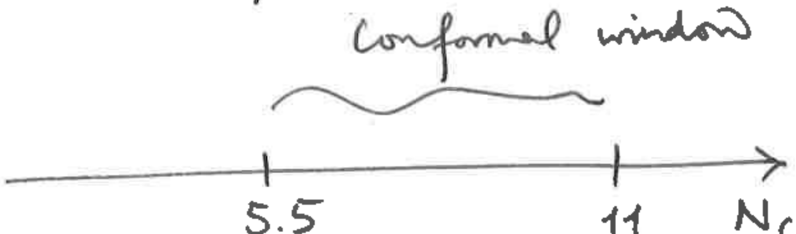
$$34\, C_A^2 < 2(5 C_A + 3 C_f) N_f$$

$$136 < \frac{98}{4} N_F$$

$$\boxed{n_F > 5.55}$$

✓

So, when $11 > n_F > 5.55$, β- function can develop a zero away from origin.



✓

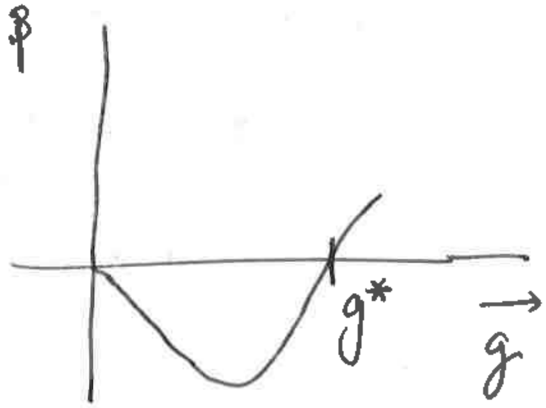
Sketch β-f^n for $N_f = 0$ & $N_f > 11$

for $N_f = 0$

$b_1 = \frac{11\, C_A}{3} = \frac{22}{3}$; $b_2 = \left(\frac{34}{3}\right) 4 = \frac{136}{3}$

It is interesting to note that there are no zeros of β - fn away from origin in either of them since that happens for $5.5 < N_f < 11$

For ex: for $N_f = 8$



c) $$\beta(\alpha) = \frac{-b_0 \alpha^2}{4\pi} + \frac{b_1 \alpha^3}{16\pi^2}$$

$$\beta(\alpha^*) = 0$$

$$\Rightarrow \frac{b_0 \alpha^2}{4\pi} = \frac{b_1 \alpha^3}{16\pi^2} \quad \alpha \; 4\pi$$

$$\frac{b_0}{b_1} = \frac{\alpha}{4\pi}$$

$$\frac{(11 C_A - 2N_f)}{34C_A^2 - 10C_A n_F - 6C_F n_F} = \frac{\alpha}{4\pi}$$

$$\frac{22 - 2n_F}{136 - 20n_F - \frac{18}{4} n_F} = \frac{\alpha}{4\pi}$$

From part ⓐ we see that if $\mu \to 0$

$$\alpha\Big|_{\mu=0} \longrightarrow \text{some constant value.}$$

and β - fn is zero..

This is the case of IR fixed point. The theory behaves in a quasi-conformal way. Since, we don't have any notion of mass scale (conformal). This IR fixed point is also known as Banks-Zaks fixed point.

A/50,

$$Q^2 A_\mu^a = \mathcal{D}_\mu^{ab}(Qc^b) - \frac{1}{2} g f^{abc} (\partial_\mu c^c c^b)$$

$$- \frac{1}{2} g^2 f^{bdc} f^{cae} A_\mu^e c^d c^b \quad \cdots ③$$

Note that since f^{abc} is anti-symmetric, we can recast $(\partial_\mu c^{[c}) c^{b]}$ as

$$(\partial_\mu c^c) c^b = \frac{1}{2} \partial_\mu (c^c c^b) \quad ✓ \quad \cdots ④$$

Using $Qc^b = -\frac{1}{2} g f^{bca} c^c c^a$ in ③ gives

$(* f^{abc} = -f^{bac})$

$$Q^2 A_\mu^a = \frac{1}{2} g f^{abc} (\partial_\mu c^c c^b) - \frac{1}{2} g f^{abc} (\partial_\mu c^c c^b)$$

$$+ \frac{1}{2} g^2 f^{bdc} f^{cae} A_\mu^e c^d c^b$$

$$- \frac{1}{2} g^2 f^{bdc} f^{cae} A_\mu^e c^d c^b$$

(Note: relabelling of indices in the underlined term)

$$= 0 \quad ✓$$

3) $\gamma_A = \frac{1}{2} \frac{\partial \ln Z_3}{\partial \ln \mu}$ ... ①

where Z_3 in $\overline{MS}$ scheme is given by:

$$Z_3 = 1 + \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + \mathcal{O}(g^4)$$

Rewrite ① as

$$\gamma_A = \frac{1}{2} \frac{\partial \ln Z_3}{\partial g} \frac{\partial g}{\partial \ln \mu} \quad \ldots \quad ②$$

We derived the expression for $\frac{\partial g}{\partial \ln \mu}$ to $\mathcal{O}(\varepsilon)$ as

$$\frac{\partial g}{\partial \ln \mu} = \frac{-\varepsilon g}{2}$$

$$\frac{\partial \ln Z_3}{\partial g} = \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] \frac{2g}{8\pi^2 \varepsilon} + \mathcal{O}(g^4)$$

$\ln(1+x) \sim x$ to $\mathcal{O}(g^2)$

Then, Eq. ② becomes

$$\gamma_A = \frac{1}{2} \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] \frac{2g}{8\pi^2 \varepsilon} \cdot \frac{-\varepsilon g}{2}$$

$$= \frac{-g^2}{16\pi^2} \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] + \mathcal{O}(g^4)$$

k solution discussed in class :

$$\varphi = v \tanh\left(\frac{m}{2}(x - x_0)\right)$$

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april 2, 2015

Lorentz boost solution (i.e moving soliton) is :

$$\varphi(x,t) = v \tanh\left[\frac{\gamma m}{2}(x - x_0 - \beta t)\right] \quad \cdots \text{①}$$

Eq^n of motion reads :

$$\Box\varphi(x,t) + V'(\varphi(x,t)) = 0 \quad \cdots \text{②}$$

where $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$

Signature used in problem $(+ - - -)$

$$\Box\varphi(x,t) = \frac{\beta^2}{(1-\beta^2)}\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right) - \frac{1}{(1-\beta^2)}\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)$$

$$= \left(\frac{\beta^2 - 1}{1-\beta^2}\right)\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)$$

$$= -\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right) \quad \cdots \text{③}$$

Note that I've set $x_0 \to 0$ & $\frac{m}{2} = 1$ to keep argument simple. They won't affect what we set to prove through the entire problem.

Now we note that

$$\Rightarrow \frac{1}{2}\left(\frac{1+\beta^2}{1-\beta^2}\right) + \frac{1}{2}$$

$$= \frac{1}{1-\beta^2}$$

Hence,

$$T(x,t) = \frac{1}{1-\beta^2}\left(\varphi'\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)\right)^2$$

↓ En. density

$$\text{Energy of moving soliton} = \int_{-\infty}^{\infty} T(x,t)\,dx$$

$$= \frac{1}{1-\beta^2}\int_{-\infty}^{\infty} \varphi'\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)^2 dx$$

Now let $\frac{x-\beta t}{\sqrt{1-\beta^2}} = \xi$

$$dx = \sqrt{1-\beta^2}\,d\xi$$

$$= \frac{1}{\sqrt{1-\beta^2}}\int_{-\infty}^{\infty} \varphi'(\xi)^2\,d\xi$$

↓ Rest Energy ✓

$$= \gamma E_{\text{rest energy}}$$

$$= \gamma M$$

2. Let $\Gamma(g) = \int d^3x \; \varepsilon^{ijk} \, \mathrm{Tr}\left\{ (g\partial_i g^{+})(g\partial_j g^{+})(g\partial_k g^{+}) \right\}$

$$= \int d^3x \; \mathrm{Tr}\,(g\partial g^{+})^3 \quad \cdots \quad (1)$$

We've to show that (1) is invariant under smooth deformations.

Sol^n: As a digression, let's prove a result useful later on.

$$\Gamma(g_1 g_2) = \Gamma(g_1) + \Gamma(g_2) \quad \cdots \quad (2)$$

Proof:- $(g_1 g_2)\partial(g_1 g_2)^{\dagger} = g_1 g_2 \partial g_2^{+} g_1^{+} + g_1 g_2 g_2^{+} \partial g_1^{+}$

$$= g_1 \left(g_2 \partial g_2^{+} + \partial g_1^{+} g_1 \right) g_1^{+}$$

We used the fact $gg^{+} = \mathbb{1}$

Infact, $g_1 g_1^{+} = g_2 g_2^{+} = \mathbb{1}$

Let $g_2 \partial g_2^{+} = X$

and, $\partial g_1^{+} g_1 = Y$

$$\Gamma(g_1 g_2) = \int \mathrm{Tr}\,(X+Y)(X+Y)(X+Y)\, d^3x$$

Now, we can show that $\Gamma(g)$ is invariant under $g \rightarrow g + \delta g$.

$$g \rightarrow g + \delta g$$
$$\rightarrow g\left(1 + \frac{\delta g}{g}\right) \quad \dots \quad (4)$$

$$\Gamma(g + \delta g) = \Gamma\left(g\left(1 + \frac{\delta g}{g}\right)\right) = \Gamma(g) + \Gamma\left(1 + \frac{\delta g}{g}\right)$$

$$= \Gamma(g) + \underbrace{\Gamma\left(1 + \frac{\delta g}{g}\right)}_{\text{zero}}$$

But $\delta g \rightarrow 0$ at $x = \pm\infty$ and $\Gamma(1) = 0$

$$\Rightarrow \Gamma(g + \delta g) = \Gamma(g) \checkmark.$$

Co-ordinate transformation independent:

Since $\Gamma(g)$ is invariant under $g + \delta g$, we can argue that it will also be invariant under $x + \delta x$ since we can think of 'δg' deformation as inducing some change in coordinate (since $g(x)$).

✓.

2) Let's write down the variation of $\delta(g\partial_i g^+)$

$$\delta(g\partial_i g^+) = \delta g\, \partial_i g^+ + g\,\partial_i \delta g^+$$

$$= \delta g(\partial_i g^+) + g\,\partial_i\left(-g^+ \delta g\, g^+\right)$$

$$= \delta g\,(\partial_i g^+) - g\,\partial_i g^+ \delta g\, g^+ - g g^+ \partial_i \delta g\, g^+ - g g^+ \delta g\, \partial_i g^+$$

$$\left\{\begin{array}{l} \delta(gg^+) = 0 \\ g\delta g^+ = -\delta g\, g^+ \\ \delta g^+ = -g^+\delta g\, g^+ \end{array}\right. \quad \cdots \text{(A)}$$

$$= \delta g\,(\partial_i g^+) - g\,\partial_i g^+\delta g\, g^+ - g g^+ \partial_i \delta g\, g^+ - \delta g\,\partial_i g^+$$

$$= -g\left(\partial_i g^+ \delta g + g^+ \partial_i \delta g\right) g^+$$

$$= -g\,\partial_i\left(g^+\delta g\right) g^+ \quad \cdots \text{(1)}$$

Let's write out $\varepsilon^{ijk}\,\mathrm{Tr}\left[(g\partial_i g^+)(g\partial_j g^+)(g\partial_k g^+)\right] = Z$ explicitly now:

$$\delta Z = \varepsilon^{ijk}\,\mathrm{Tr}\left[(g\partial_i g^+)(g\partial_j g^+)\,\delta(g\partial_k g^+)\right]$$

$$= -\varepsilon^{ijk}\,\mathrm{Tr}\left[(g\partial_i g^+)(g\partial_j g^+)\,g\,\partial_k(g^+\delta g)\,g^+\right] \quad \text{(using (1))}$$

As written earlier, this is clearly also invariant under coordinate change of form $x \longrightarrow x + \delta x$ because we could think of $g \to g + \delta g$ (which we proved invariant) as inducing some coordinate change, as well.

3) a $\langle \theta' | H | \theta \rangle = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-im\theta'} \langle m | H | n \rangle$

where $\langle m | H | n \rangle \sim e^{-|n-m| S_0}$

$$= \sum_{\substack{m=-\infty \\ m=n}}^{\infty} e^{im\theta} e^{-im\theta'} \langle m | H | m \rangle + \phi$$

$$= \sum_{m} e^{im(\theta-\theta')} e^{0} + \phi$$

$$= \delta(\theta-\theta') + \sum_{m} \sum_{n} e^{in\theta} e^{-im\theta'} \langle m | H | n \rangle$$

Now, in the second term, we can have $m = n \pm 1$, $m = n \pm 2$ and so on.. but we can always change the limits of summation to ensure we pick a δ-function

$$= \delta(\theta-\theta') \left[1 + e^{-s} \cos\theta + e^{-2s} \cos 2\theta + \cdots \right]$$

Note that since $S_0 \gg 1$, terms apart from $m = n$ will be i.e $e^{-|m-n| S_0} \ll 1$ when $m \neq n$, and vanish as difference increases.

Hence,

$$\langle \theta' | H | \theta \rangle = \delta(\theta-\theta') \left[\text{Some Energy Eigenvalue for ex: } 1 + e^{-s} \cos\theta + \cdots \right]$$

✓.

3. a. $\langle \theta' | H | \theta \rangle = \sum_{m,n} e^{-im\theta'} e^{+in\theta} \langle m | H | n \rangle$

Let $n \to m + \xi$

$$= \sum e^{im\theta} e^{i\xi\theta} e^{-im\theta'} f(m-n)$$

$$= \sum_m e^{im(\theta-\theta')} \sum_\xi e^{i\xi\theta} f(-\xi)$$

$$= \delta(\theta-\theta') \; f(\theta)$$

where $f(\theta) = \sum_\xi e^{i\xi\theta} f(-\xi)$ is the energy eigenvalue

Hence $\langle \theta' | H | \theta \rangle \propto \delta(\theta - \theta')$ ✓ — ①

b) We note that $e^{-H\tau} \approx \mathbb{1} - H\tau + \dots$ where $H\tau \ll 1$

Let's replace H by $\mathbb{1}$ in ①

$$\langle \theta' | \mathbb{I} | \theta \rangle \propto \delta(\theta - \theta')$$

$$\Rightarrow \approx \langle \theta' | e^{-H\tau} | \theta \rangle \propto \delta(\theta - \theta')$$

We can calculate average value of $e^{-H\tau}$ over all θ-vacua.

Comparing with L.H.S, we see that

$$H\tau = -2e^{-S_0}\cos\theta$$

$$\boxed{H \sim -\cos\theta}$$

Note : I could never have figured this negative sign without useful reference found on John Preskill's CalTech website !