Notes on large N and String Theory

Notes for the HEP group meetings

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These notes are based on author's understanding. It may contain errors and should be used with caution.

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Contents

“Large N is a life long obsession“

0.1 Introduction

Some quick remarks :

- The sphere (or the plane) has the largest Euler characteristic, $\chi = 2$. Planar can be drawn on plane (without crossing), also called "non-crossing".
- In QED, the expansion parameter is $e^2/4\pi = 1/137$ which implies that $e = 0.3$. Therefore, one cannot just rule out the possibility of using $1/N$ in QCD.

Very often, we associate more variables with greater complexity. But this is not always true. There are class of theories which simplifies in the large N limit. Certain field theories with symmetry group $SO(N), SU(N), U(N)$ simplify in the large N limit.

One might think that a reasonable expansion parameter for QCD can be g_{YM}, but this is not quite true in the light of the renormalization group equations. QCD has no obvious free parameter and this makes things very difficult to calculate in perturbation theory.

The large N limit exists for vector as well as matrix models. QCD is an example of matrix model because the fields are $N \times N$ matrix (like A_μ^{ij}) which has N^2 components. Since the matrix must be traceless, it should have one less component, but the difference is not important in the large N limit The vector models for example only have N components.

0.1.1 Large N limit for vector models

TO ADD !!! (but not now)

0.2 large N limit

In the Standard Model of elementary particle physics, strong interactions are described by QCD – a non-Abelian gauge theory based on SU(3) color gauge group. To understand how we can think of large N limit of QCD, let's look at the two-loop perturbative QCD β-function given by,

$$\mu \frac{dg}{d\mu} = -\frac{1}{4\pi^2}\left(\frac{11N - 2n_f}{3}\right) g^3 - \frac{1}{4\pi^4}\left(\frac{34N^3 - 13N^2 n_f + 3nf}{3N}\right) g^5 + \mathscr{O}(g^7) \tag{1}$$

This has no sensible large N limit. But we can make an interesting observation. The LHS of (1) goes as $\sim g$, whereas RHS $\sim g^3 N$. 't Hooft considered the limit where $N \to \infty$ and $g^2 \to 0$ while $\lambda = g^2 N$ remains fixed. λ is the 't Hooft coupling. In this limit, we get,

$$\mu \frac{d\lambda}{d\mu} = -\frac{11}{24\pi^2}\lambda^2 - \frac{17}{192\pi^4}\lambda^3 + \mathscr{O}(\lambda^4) \tag{2}$$

As can be readily seen, perturbation theory predicts that the 't Hooft limit of QCD is an asymptotically free theory. So, that's good first step. It is also natural to assume that the $\Lambda_{\rm QCD}$, the scale parameter of strong interactions is held fixed as $N \to \infty$. One important feature of 2 is that it is independent of number of flavors, n_f. This can be understood as follows : the number of quark

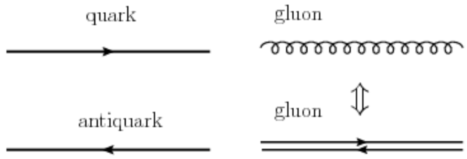

Figure 1: –

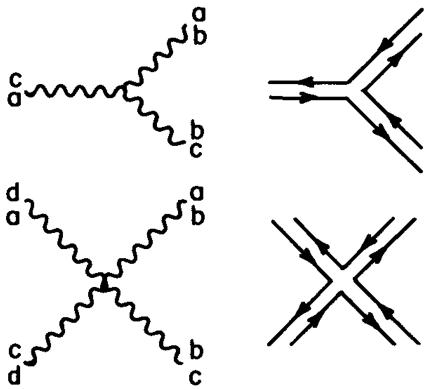

Figure 2: Both diagrams contribute $\sim N/\lambda$. Note that E = F = 0 and V = 1 for both.

degrees of freedom is $\mathscr{O}(nfN)$, i.e. $\mathscr{O}$(N) in the 't Hooft limit, and hence sub-leading with respect to the number of gluon degrees of freedom, which is $\mathscr{O}\,(N^2)$.

Let's consider the Lagrangian (actually, modified version of Yang-Mills Lagrangian, and also incomplete)

$$\mathscr{L} \sim \frac{1}{g_{\mathrm{YM}}^2}\left[-\frac{1}{4}F^a_{\mu\nu b}F^{\mu\nu b}_a + \text{fermions}\right], \tag{3}$$

with $F^a_{\mu\nu b} = \partial_\mu A^a_{\nu b} - \partial_\nu A^a_{\mu b} + i[A_\mu, A_\nu]^a_b$.

The 't Hooft limit is the limit where $N \to \infty$ and $g^2_{\mathrm{YM}} \to 0$ while $\lambda = g^2_{\mathrm{YM}}N$ remains fixed. λ is the 't Hooft coupling. Also, the advantage for using this particular form of the Lagrangian is that any vertex has same factor g^2_{YM}. We need not worry about the difference between three-gluon, four-gluon vertex contributing different factors. So, both diagrams in Fig(2) contribute the same N/λ.

With our current normalization [2] the propagator $\langle\Phi\Phi\rangle$ goes like λ/N. The propagator in the double line notation appears in Fig. 1. We can naively see that the gluon propagator and quark

[2]We should be careful about field normalizations when doing large N counting for correlation functions.

propagator $\sim 1/N$ whereas, the vertex $\sim N$ in the 't Hooft limit.

We can write the gluon propagator as (shown in Fig 1),

$$A^a_{\mu b}(x) A^c_{\nu d}(y) = \left(\delta^a_d \delta^c_b - \frac{1}{N}\delta^a_b \delta^c_d\right) \mathscr{D}_{\mu\nu}(x-y), \tag{4}$$

In some sense, the gauge field is represented by a "quark" with index i, and an "antiquark" with index j.

The second term in the parantheses is because we need to make the gluon field traceless for SU(N) group we are considering. It would not be present if we were working with U(N). In any case, in the large N limit, the distinction is unimportant.

Following the gluon line indices we see, the index pair at the beginning is same as that at end. In some sense, a gluon propagates like quark–anti-quark pair. This observation was made by 't Hooft when he devised the double-line notation. Loosely speaking, we will draw as many lines in a propagator as the indices it carries. Therefore, quark propagator is denoted by single line since it carries one index, whereas a gluon propagator is drawn using double lines (see Fig(1)).

For SO(N) or USp(N) theories, the adjoint representation may be written as a product of two fundamental representations rather than a product of a fundamental and an anti-fundamental representation (like done for SU(N) and U(N)). Since the fundamental representation is real, there are no arrows on the propagators [3].

Generally, a vacuum diagram has the following dependence on g^2_{YM} and N,

$$\text{Amplitude} \sim \text{diagram} \sim (g^2_{\mathrm{YM}})^E (g^2_{\mathrm{YM}})^{-V} N^F$$

where E is the number of propagators, V is the number of vertices, F is the number of faces. This has no sensible $N \to \infty$ limit, since there is no upper limit on F. However, 't Hooft suggested that we can take the limit $N \to \infty$ and $g^2_{\mathrm{YM}} \to 0$ but keep $\lambda = g^2_{\mathrm{YM}} N$ remains fixed.

$$\text{diagram}(V, E, F) \sim N^{V-E+F} \lambda^{E-V} \sim N^{\chi} \lambda^{E-V}$$

where, $\chi = F + V - E$ is assured by a theorem due to Euler.

Theorem: Given a surface composed of polygons with F faces, E edges and V vertices, the Euler character satisfy $\chi = F + V - E = 2 - 2h$. here, h is the number of handles (also known as 'genus') of the surface. See Fig(3) for example. Since, in the large N limit, the diagrams with $h = 0$ contribute most [4] , the large N limit is also known as planar limit (because $h = 0$ means no handles like spheres)

[3]Recall that arrows fix the orientation of complex fields

[4]Every graph that can be drawn on a sphere can be drawn on the plane as well, and vice versa. We can also think

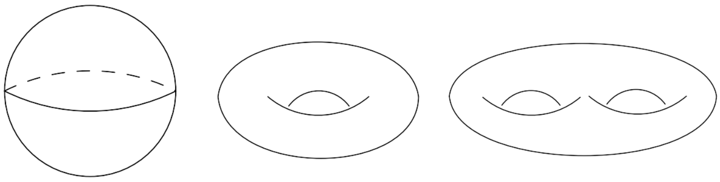

Figure 3: Three surface with $h = 0, 1$ and 2 respectively.

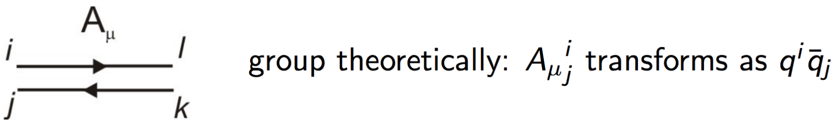

Figure 4: Representing —

Since each Feynman diagram can be considered as a partition of the surface separating it into polygons, then the above theorem also works for our counting in N. Only planar diagrams survive in the large N limit.

As an example, let's consider the diagram shown in Fig (5), which has four 3-point vertices, six propagators, and four index (color) loops. [5]

as follows: Remove one randomly selected face from the spherical surface and project the the remainder of the surface onto a plane.

[5] Prof. McGreevy's Lec-7, Lec-8

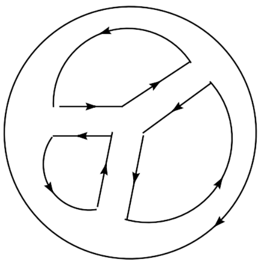

Figure 5: A diagram with E=6, V=4 and F=4 which contributes $\sim N^2\lambda^2$.

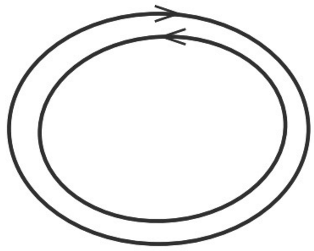

Figure 6: A diagram with E=0, V=0 and F=2 which contributes $\sim N^2$.

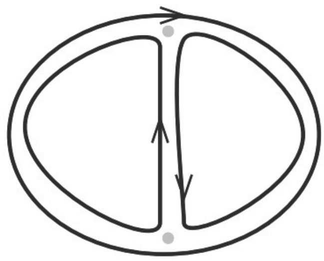

Figure 7: A diagram with E=3, V=2 and F=3 which contributes $\sim N^2\lambda$.

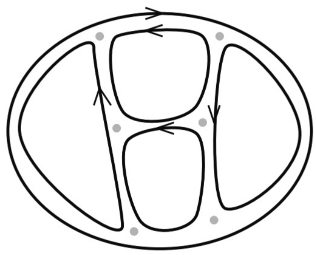

Figure 8: A diagram with E=9, V=6 and F=5 which contributes $\sim N^2\lambda^3$.

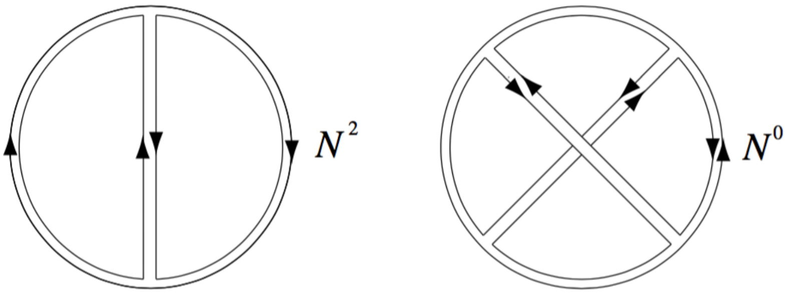

Figure 9: Planar and non-planar diagrams. The diagram on the left is same as (7) and contributes $\sim N^2\lambda$. The diagram on the right is slightly complicated. It has E =6, F=2, V=4 and contributes $\sim N^0\lambda^2$. It has one outer color loop and one tangled color loop which cannot be drawn in a plane.

$$\log \mathscr{Z} = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda) \tag{5}$$

$$= N^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \cdots \tag{6}$$

The first term comes from the planar diagrams, second term from the genus-1 diagrams, and so on... Hence, in the large N limit, the free energy goes as $\mathscr{O}(N^2)$.

One might think that the free energy diverges in the large N limit, but we actually calculate,

$$\lim_{N\to\infty} \frac{F}{N^2} = \cdots$$

which has a sensible large N limit.

Another way to understand that $\log \mathscr{Z} \sim \mathscr{O}(N^2)$, is to look at the partition function,

$$\mathscr{Z} = \int \mathcal{D}\Phi \exp^{iS[\phi]}$$

, with

$$L = \frac{N}{\lambda} \mathrm{Tr} \left(\frac{1}{2}(D\Phi)^2 + \frac{1}{4}\Phi^4 \right)$$

and since the fields are matrix valued, we have $L \sim \log \mathscr{Z} \sim \mathscr{O}(N^2)$.

General observables we consider are correlation functions of gauge invariant operators,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{\text{con}} \tag{7}$$

we will assume that $\mathcal{O}$ is a single trace operator. it is enough to just consider them since the multiple trace operator are just products of them.

- Single-trace operators : $\text{Tr}\ (F_{\mu\nu}F^{\mu\nu}), \text{Tr}\ (\Phi^n)$
- Double-trace operators : $\text{Tr}\ (F_{\mu\nu}F^{\mu\nu})\text{Tr}\ (\Phi^2)$

Generally, single trace operators are like,

$$\mathcal{O}(x) = \text{Tr}\ (\Phi_1(x)\cdots\Phi_k(x))$$

which we can denote by $\text{Tr}\ (\Phi_k(x))$

Our aim is to understand how does (7) behaves in the large N limit [6]. Consider (not worrying about factors of i, π, etc..)

$$\mathscr{Z}[J_1,\cdots J_n] = \int \mathcal{D}A\mathcal{D}\Phi\cdots \exp\left[S_0 + N\sum_j \int J_i(x)\mathcal{O}_i(x)\right] \tag{8}$$

Then, (7) can be written as,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{\text{con}} = \lim_{\text{all } J\to 0} \frac{\delta^n \log\mathscr{Z}}{\delta J_1(x_1)\cdots\delta J_n(x_n)}\frac{1}{N^n} \tag{9}$$

But, we know that

$$\log\mathscr{Z}[J_1\cdots J_n] = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda,\cdots) \tag{10}$$

So, we get,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_{\text{con}} \sim N^{2-n}\left[1 + \mathscr{O}(\frac{1}{N^2})\right] \tag{11}$$

and it implies, for ex.

$$\langle I\rangle \sim \mathscr{O}(N^2) + \mathscr{O}(N^0)$$

$$\langle \mathcal{O}_1\mathcal{O}_2\rangle_{\text{con}} \sim \mathscr{O}(N^0) + \mathscr{O}(N^{-2})$$

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{\text{con}} \sim \mathscr{O}(N^{-1}) + \mathscr{O}(N^{-3})$$

[6] I learned about this from Prof. Hong Liu lectures at MIT. Available here

Suppose that $\langle\mathcal{O}\rangle \sim \mathscr{O}(1)$. Then the variance of $\langle\mathcal{O}\rangle$ is : $\langle\mathcal{O}^2\rangle$ - $\langle\mathcal{O}\rangle^2 = \langle\mathcal{O}^2\rangle_{\text{con}} \sim \mathscr{O}(1/N^2)$.
This implies, for two gauge-invariant operators, A and B (in the large N limit)

$$\langle AB\rangle \sim \langle A\rangle\langle B\rangle + \mathscr{O}(\frac{1}{N^2})$$

Variance of operators vanish in this limit and there are no fluctuations. All this has been done for the case of pure gauge theory but this can be extended to fermions as well. To summarize, *The average value is the only calculated value via the master field*, about which we now discuss.

Another immediate consequence of this can be applied to Wilson loop as done by Migdal in 1977. We can show that the following has,

$$\phi(\mathcal{C}) = \frac{1}{N}\text{Tr}\left(\mathscr{P}\exp\oint_{\mathcal{C}} A_\mu dx^\mu\right) \tag{12}$$

a decoupling property in the large N limit,

$$\langle\phi(\mathcal{C}_1)\ \phi(\mathcal{C}_2)\rangle \to \langle\phi(\mathcal{C}_1)\rangle\langle\phi(\mathcal{C}_2)\rangle \tag{13}$$

The above equation 13 implies that $\phi(\mathcal{C})$ can be considered as a classical field in loop space, but it does not imply that the gauge field, A_μ become classical in this limit.

0.3 Master field

Large N limit is also tied to the idea of a master field (coined by Coleman) initially due to Witten. In this limit, the probability of finding any gauge-invariant quantity away from its expectation value goes to zero as $N \to \infty$. The best way to think of master field is to draw analogy with the path-integral approach. It tells us that Green's functions for a quantum theory are obtained by summing over all possible classical motions. In the limit that $\hbar \to 0$, the measure in the functional space becomes sharply concentrated about the solution to the classical equations of motion, in the limit of vanishing $\hbar$, all quantities are given by their value evaluated at the classical solution. In the case of large N limit of a gauge theory, there exists a master field configuration. All gauge-invariant operators expectation value can be evaluated using this master field. In case of large N QCD, the master field(s) are "*four* $\infty \times \infty$" M_i matrices !! Some observations,

- We can calculate all the correlation functions of the invariant observables simply by taking trace of the product of master fields and not doing any integrals.

- The master field is not unique, since we are interested only in gauge-invariant, *any gauge transform of a master field is also a master field.* So more precisely, we should say 'master orbit of the gauge group'.

- For the classical analogy, we have a well-defined method of finding the single field configuration, i.e by solving classical equations of motion. But finding the master field is not a panacea, one does not have a well-defined prescription yet.

This is an amazing reformulation of the problem of summing only all planar graphs.

0.4 Origins

In the 1960s, theory of strong interactions was undergoing major paradigm shift. In particular, lot of experimental data was coming in from the scattering experiments of mesons et. al and there was a curious behavior. The angular momentum of the mesons was always proportional to their masses. One could fit a straight line with a constant slope which was also known as the universal 'Regge-slope'. Experimental data also gave rise to another curious phenomenon. In QFT, one adds contributions from various channels to calculate the scattering matrix. In scattering of mesons, it was observed that the contribution of s-channel was exactly equal to that of t-channel. One was given an option of choosing either rather than the usual addition as demanded by the rules of QFT. We will switch between excited states and resonances intermittently. Dual resonance models inspired from Veneziano amplitude were seen as a promising alternative to QCD (Quantum Chromodynamics) as theory of strong interactions. Dual resonance models were able to explain the Regge trajectory and also this s-t symmetry.

0.5 Conditions on spinors

A vector field A^μ in D=10 has ten components, but only eight transverse components describe independent propagating modes (d-1-1). Supersymmetry requires that massless spinor should also have eight propagating modes. In general, a spinor in 10-d has $2^5 = 32$ complex components. But, we will see that imposing both the Majorana and Weyl condition will reduce this to eight complex or sixteen real components. These remaining components must still satisfy :

$$\Gamma\partial\chi = 0 \tag{14}$$

The above Dirac equation relates one half to the other. Thus, the number of propagating modes described by the spinor satisfying a Dirac equation is half the number of components that the spinor contains. Majorana-Weyl in ten dimensions has eight propagating modes, the number required to form supermultiplet with A_μ. Dirac spinor in 10 dimensions has 32 complex, Majorana reduces it to 16 complex, Weyl reduces it to 8 complex or 16 real spinors. When we dimensionally reduce to d=4, it still has 16 real components but in 4d, the Majorana-Weyl condition is not valid simultaneously. The N = 4 action has four Weyl spinors, each with two complex (four real) degrees of freedom. In d=10, one had a Dirac spinor with 32 complex components, in 4d- one has four Dirac spinors each with 4 complex(8 real) components. A Dirac spinor in 4-d has $2^3 = 8$ real components. Four fermions of the N=4 SYM will have 32 real components. If we consider either Weyl or Majorana fermion in 4-d, it will have 16 real components. It can be repackaged as a single Weyl-Majorana fermion in ten dimensions which has 16 real components. In d dimensions, these spinors have 2[d/2] complex components, but this representation is not irreducible. For an irrreducible representation, we must impose either the Weyl (chirality) condition, or the Majorana (reality) condition, or in some cases both.

0.5.1 Majorana condition

This condition is also known as reality of fermion field. The representation of Γ's in which the Ψ's are real is Majorana condition. The real spinors are also called as 'Majorana spinors' after famous Italian physicist Ettore Majorana.

0.5.2 Weyl condition

If D is odd, the Lie algebra representation is irreducible. If D is even, it splits further into two irreducible representations - called the Weyl or half-spin representations. In case when D is even, like in our 4-d world, once can define a gamma matrix γ^5 which denotes the chirality of the spinors. In general, in D=10, it can be defined as :

$$\Gamma_{11} = \Gamma 0.....\Gamma 9 \tag{15}$$

This satisfies,

$$\Gamma_{11}, \Gamma_{\mu} = 0$$

and,

$$\Gamma_{11}^2 = 0$$

Spinor of definite chirality is called Weyl-spinor. In Majorana representation, if the ten Γ's are imaginary, then it is clear that the Γ_{11} is real. Therefore, given a real spinor χ, the two pieces of definite chirality are also real. This means that D=10 satisfies both Majorana and Weyl condition. In D=4, they are not compatible since in Majorana representation, γ_5 is imaginary. In general, they are compatible only in D=2(mod 8). It is because of this, that, superstring theory necessarily needs ten dimensions to propagate correct number of modes.

0.6 Modified Feynman diagram and q, $\bar{q}$

Suppose a meson consists of two quarks rotating around a center of mass. What force law could reproduce the simple behavior of the equation mentioned below ? Diagrammatically, this can be seen as follows : Let us now consider the following replacement to the Feynman diagram first thought (probably) by Nambu & Susskind.

$$J = \alpha(s) = \alpha_0(s) + \alpha^{'}(s) \tag{16}$$

where s $= M^2$ also known as COM energy.

Assume that the quarks are moving at speed c (which is reasonable, because most of the resonances are much heavier than the lightest, the pion). Let the distance between the quarks be r. Each has a transverse momentum p. Ignoring the energy of the force fields themselves (and put c $=$ 1)

$$s = M^2 = (2p)^2 \tag{17}$$

Also $J = \mathrm{p} \cdot \mathrm{r}$ and centripetal force $= \frac{pc}{r/2} = 2p/r$

This gives us the relation :

$$F = \frac{s}{2J} = \frac{1}{2\alpha'}$$

Of course, we all know that this interpretation is wrong since we cannot ignore the contribution of the force field etc. etc.

Let's consider a more complicated model of a massless and spinless quark connected by a rigid rod of length $2r_0$. This rod which can be thought of as a rigid string mimics the flux tube that connects the quarks together in a meson having a tension or energy per unit length σ. For a given length, the maximum angular momentum J is observed when the ends are moving with the velocity of light (here, c=1) The velocity of a particle at a distance 'r' from center will be $v(r) = r/r_0$, the total mass of the system is given by :

$$M = 2\int_0^{r_0} \frac{\sigma}{\sqrt{1 - v(r)^2}} dr = 2\sigma \int_0^{r_0} \frac{\sigma}{\sqrt{1 - (r/r_0)^2}} dr = \sigma\pi r_0 \tag{18}$$

And the angular momentum of the string is :

$$J = s\int_0^{r_0} \frac{\sigma r v}{\sqrt{1-v(r)^2}}dr = \frac{2\sigma}{r_0}\frac{r^2 dr}{\sqrt{1-(r/r_0)^2}} = \sigma\frac{\pi}{2}r_0^2 \tag{19}$$

From (3) and (4), we conclude that :

$$J = \frac{M^2}{2\pi\sigma} \tag{20}$$

Baryons and mesons were observed to have angular momentum J, satisfying the following equation :

$$J = \alpha_0 + \alpha^{'} M^2 \tag{21}$$

where $\alpha^{'}$ is called the universal Regge slope. It is of the order $1GeV^{-2}$. This slope $\alpha^{'}$ is the only free paramter in the string theory snd is related to the tension of the string as :

$$\sigma = \frac{1}{2\pi\alpha^{'}} = T$$

0.6.1 BPS States

BPS states are important tool in establishing dualities concerning the strong-weak coupling regimes. Most of the quantitative knowledge we have about quantum field or string theory is the result of a weak coupling (perturbative analysis) and needless to say, an extrapolation to the strong coupling is usually not tractable. In particular, results derived from the classical action can be completely spoiled by large quantum effects. But, there are examples where the extrapolation to strong coupling regime can be reliably performed and this involves BPS states of supersymmetric theories. Their mass is determined by their charges and this relation, being a result of the representation theory of the supersymmetry algebra (Remember : Q, Q Z), must be true for all values of the coupling constant unless several short multiplets combine into longer multiplets which can violate the BPS bound. If the charge is quantized, the mass of BPS states cannot change continuously. In string theory such quantized charges can be momenta and winding numbers in compact directions. Therefore, once the BPS states are identified in the classical approximation of some superstring theory, they should exist also in the non-perturbative regime. This is the reason of why they are interesting.

Dimensions(d)	**Majorana**	**Weyl**	**Weyl-Majorana**	**Min.Rep**
2	Yes	Self	Yes	1
3	Yes	-	-	2
4	Yes	Complex	-	4
5	-	-	-	8
6	-	Self	-	8
7	-	-	-	16
8	Yes	Complex	-	16
9	Yes	-	-	16
10 (*mod 8*, 2)	Yes	Self	Yes	16
11 (*mod 8*, 3)	Yes	-	-	32
12 (*mod 8*, 4)	Yes	Complex	-	64

Table 1: *Dimensions in which various conditions are allowed for SO(d-1,1) spinors*

0.7 Spinors in various dimensions

We will often come across notations where $\mathcal{N} = (2,2)$ and $\mathcal{N} = (8,8)$. This is different from the notation we are used to in four dimensions. In 4d, the Weyl representation is complex, so that the representation of $\bar{Q}$ is fixed to be the conjugate of the Q representation. In 2d, the Weyl representation is real (*Majorana-Weyl*) and the $\bar{Q}$ representation is independent of the Q representation. This can be seen for various dimensions from Table 1.