$$e^{Z} = \int dM\; e^{\underline{-\frac{1}{2}\mathrm{tr}\,M^2 + \frac{g}{\sqrt{N}}\mathrm{tr}\,M^3}}$$

where M is $N \otimes N$ matrix

The underlined term only depends on eigenvalues of matrix '$\underline{M}$', we can factorize the int. measure $_{(I.M)}$ 'dM' into the product of Haar measure for unitary matrices and I.M for eigenvalues.

The integration over the ~~Haar measure~~ unitary matrices is trivial, since $\int dU\; 1 = 1 \longrightarrow$ properties

$$e^{Z} = \int dM\; e^{-\mathrm{tr}\,V(M)} \qquad V(M) = M^2 + \sum_{K \geq 3} \alpha_K M^K$$

$$= \int \prod_{i=1}^{N} d\lambda_i\, \Delta^2(\lambda)\, e^{-\sum_i V(\lambda_i)} \qquad \ldots\ldots \quad (1)$$

λ_i are the ~~#~~ (N) eigenvalues of hermitian matrix M. and

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

(1) can also be derived alternatively used Fadev-Popov method. See hep-th/9306153

for $\underline{\underline{N=3}}$

$$(\lambda_3-\lambda_2)(\lambda_2-\lambda_1)(\lambda_3-\lambda_1) = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}$$

$$\Delta(\lambda) = \det\left\{ \lambda_i^{j-1} \right\}$$

If you are studying matrix models, say large-N, QCD in two dimensions & several other places, occurence of Vandermonde determinant is quite frequent

Let's modify ① and consider generally,

$$e^{Z(g,\alpha_k,N)} = \int dM\, e^{-(N/g)\,\mathrm{tr}\, V(M)}$$

$$= \int \prod_i d\lambda_i \cdot \Delta^2(\lambda)\, e^{-\frac{N}{g}\sum_i V(\lambda_i)}$$

Also note that, →

The $\underline{\text{Vandermonde}}$ determinant leads to a repulsive force between eigenvalues which otherwise would accumulate at V_{min}.

Fuzzy sphere (more about it later) arises as a vacuum solution of several matrix models.
in the large N or otherwise (?) limit.

hep-th/0307075

The simplest of matrix model is the IKKT model (hep-th/9612115)

$$S_{\substack{IKKT \\ (0+0)}} = \frac{1}{g^2} \mathrm{Tr}\left(\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Psi}^\alpha \Gamma^\mu_{\alpha\beta}[X_\mu, \Psi^\beta]\right)$$

X are $N \times N$ hermitian bosonic matrices

$\mu = 1 \cdots D$

Ψ are $N \times N$ Ψ_α with $\alpha = 1 \ldots 2^{[D/2]}$

This is the reduction to 0-dimension of $\mathcal{N}=1$ SYM theory in D-dimensions. Note that $\mathcal{N}=1$ SYM theory can only exist in $D = 10, 6, 4, 3$

In what follows, we will assume $\underline{\mathcal{D}=10}$

IKKT model ~~assumes~~ enjoys $SO(1,9)$ symmetry.

Moving on, if we move to the reduction of $N=1$ SYM for $d=10$ to one-dimensions, we end up with BFSS model. It is $SO(9)$ invariant.

The <u>BFSS</u> action is given by,

$$S = \frac{1}{g^2}\int dt \, \mathrm{Tr}\Big(-\frac{1}{4}[X_I, X_J]^2 + \frac{1}{2}(\mathcal{D}_\tau X_I)^2 - \psi^T \Gamma^I [X_I, \psi] - \psi^T D_0 \psi\Big)$$

BFSS model possesses 16 real supercharge (SUSY's) and is considered to be description of Type IIA string theory.

But, BFSS has a problem: It has flat directions (when $[X^I, X^J] = 0$ and X^I belong to the Cartan subalgebra of the gauge group $SU(N)$.

Up to gauge transformations, the matrices are diagonal : eigenvalues are actually position of D0-brane.

This matrix model describes a DLCQ quantization of M-theory in <u>FLAT SPACE</u>. The discrete momentum is the rank of the gauge group i e $N \to \infty$, (dynamics at infinite momentum. Weinberg 1966)

Note that the scalars (X) can diverge along the flat directions and possess <u>large eigenvalues</u>. This implies that the bound state of the black hole is lost since the $\underline{\underline{N}}$ eigenvalues

denotes the position of N- D0 branes.
Hence; the Euclidean partition f^a is ill- defined. This problem is fixed in BMN model.

This model has been well- studied including a case where a Chern- Simons term was coupled to S_{boson} as:

$$S = \int_0^\beta d\tau \left[\frac{1}{2}(D_\tau X_i)^2 - \frac{1}{4}[X^I, X^J]^2 + i\frac{2\alpha}{3}\underbrace{\epsilon_{ijk}X_iX_jX_k}_{\boxed{\text{Myers term}}} \right.$$

BFSS has a single deconfined phase (thermal loop) corresponding to the black-hole horizon.

BMN model

Instead of considering 0+1 - dim SYM QM on flat space where we have $[X^I, X^J] = 0$ (flat directions) and no interesting phase structure one can consider a mass deformation of BFSS known as BMN or PWMM

Since the background is plane-wave matrix model
P W M M

$$S = S_{BFSS} - \frac{N}{2\lambda}\int d\tau \, \mathrm{Tr}\left[\frac{\mu^2}{9}(X^i)^2 + \frac{\mu^2}{36}(X^a)^2 + \frac{\mu}{4}\bar{\psi}\gamma^{123}\psi + i\frac{2\mu}{3}\epsilon_{ijk}X_i X_j X_k\right.$$

$i = 1..3$
$a = 4..9$

→ SO(9) broken to SO(6) ⊗ SO(3)

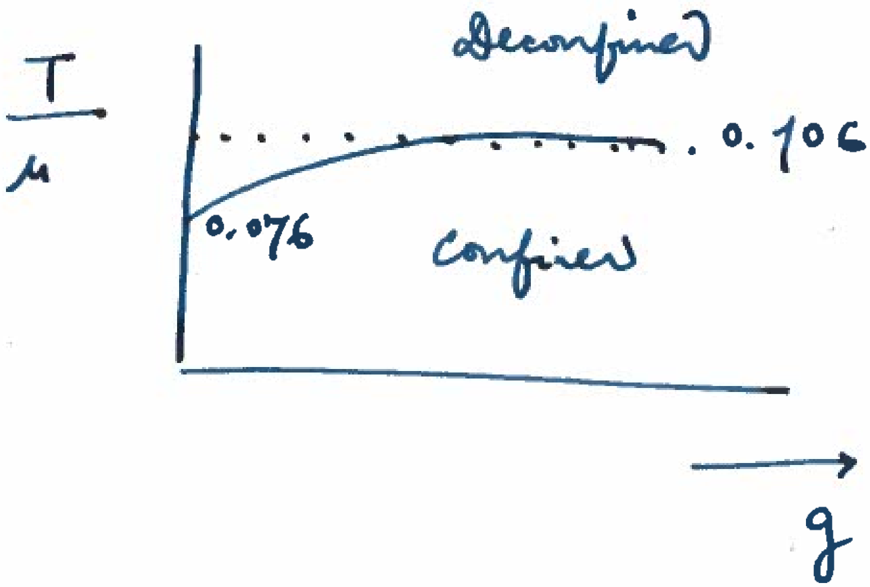
→ phase structure

→ retains 16 SUSY's

→ no flat directions

→ fuzzy sphere solutions (more about it later).

Actually, this means that, if we expand the fields around one of the minima, they are given by $X^a = 0$ and $X^i = \frac{\mu}{3} J^i$, with J^i a N- dim rep of $SU(2)$. In fact in BMN → no. of vacua equal integer of $\underline{\underline{N}}$ (?..)



Let's now discuss basics about the Fuzzy Sphere..

Fuzzy sphere

(J Madore 1991
J. Math Phys 32, 332)

To a practical physicist, non-commutative geometry (NCG) is what the name says (literally); i.e the coordinates do not commute. The first example is the quantization of the classical phase space, i.e

$$[\hat{q}, \hat{p}] = i\hbar$$

(BTW, as a digression here is a paradox:)

$$[\hat{q}, \hat{p}] = i\hbar\, \mathbb{I}_{N\times N}$$

$$\mathrm{Tr}\,[\hat{q}, \hat{p}] = i\hbar N$$

$$\underset{\downarrow}{}$$

$$0 = \boxed{i\hbar N} \;\; ?$$

Solution:- $\hat{p}, \hat{q}$ ~~are not~~ $\in$ [Trace class] "operators whose trace can be taken"

does not belong to finite-dim rep of Hilbert space

Lets take $\mathbb{R}^3$, ordinary cartesian x_i $i = 1..3$, define the sphere S^2 as the set of points in $\mathbb{R}^3$ obeying

$$x_1^2 + x_2^2 + x_3^2 = R^2 \quad \text{for some } R \in \mathbb{R} \quad \dots \quad (3)$$

In other words, all points on S^2 are at distance R from origin.

Property of sphere :-

Any smooth f^n on S^2 can be approximated to preferred (arbitrary) accuracy in the variables x_i restricted by (3).

So, all smooth 'f' on S^2

$$f(x_i) = f_0 + f^i x_i + \frac{f^{ij} x_i x_j}{2!} + \dots$$

Replace $x_i \longrightarrow \tilde{x}_i = \kappa \sigma_i$: by Pauli matrices.

$$\tilde{x}_1^2 + \tilde{x}_2^2 + x_3^2 = 3\kappa^2 = R^2$$

$$\Rightarrow \boxed{\kappa^2 = \frac{R^2}{3}}$$

Rotation in $\mathbb{R}^3$ $\qquad \uparrow \sigma$

basically all this drama is $\boxed{SO(3) \cong SU(2)}$

$$[\tilde{x}_1, \tilde{x}_2] = i\kappa \tilde{x}_3$$

we can only diagonalize one generator $\tilde{x}_i$ at a time and notion of points on the sphere is lost !! Each generator (σ) has two eigenvalues $\lambda = \pm 1$. So, we can identify north and south pole only.. FUZZY SPHERE

Note that $\kappa^2 = \dfrac{R^2}{3}$ for $SU(2)$ generators

$$\boxed{\kappa^2 = \frac{R^2}{n^2 - 1}}$$ for $SU(N)$ gen rep.

N is the dim. of representation.

In the limit $N \to \infty$, $\kappa \to 0$ $\to$ back to (not fuzzy) S^2