RAGHAV

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1) Show that the advanced propagator defined by

$$D_{adv}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ikx}}{k^2-m^2-i\,sgn(k_0)\varepsilon} \quad - (1)$$

non-zero only for $x_0 > 0$

$$Sgn(x) = +1 \quad x > 0$$
$$= -1 \quad x < 0$$

$$sgn(k_0) = +1 \text{ if } k_0 > 0$$
$$= -1 \quad k_0 < 0$$

Sol^n: Looking at the poles of (1)

$$k^2 - m^2 - i\,sgn(k_0) = 0$$

$$k_0^2 - (\vec{k}^2 + m^2) - i\,sgn(k_0)\varepsilon = 0$$

$$\Rightarrow k_0^2 = \underbrace{\vec{k}^2 + m^2}_{\omega_k^2} + i\,sgn(k_0)\varepsilon$$

$$= \omega_k^2 + i\,sgn(k_0)\varepsilon$$

$$k_0 = \pm\sqrt{\omega_k^2 + i\,sgn(k_0)\varepsilon}$$

$$= \pm\sqrt{\omega_k^2 \pm i\varepsilon}$$

$$\simeq \cancel{\pm}\left(\omega_k \pm \frac{i\varepsilon}{2\omega_k}\right)$$

$$\simeq \pm\omega_k + i\varepsilon' \qquad \left(\varepsilon' = \frac{\varepsilon}{2\omega_k}\right)_{\varepsilon' > 0} ;\ x_0 > 0$$

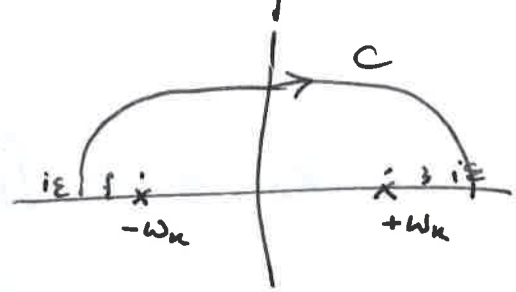
The two poles are in the upper half of the complex plane hitting the axes when $\epsilon \to 0$.

So,

$$D_{adv}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik^0 x_0}}{k^2 - m^2 - i\,\mathrm{sgn}(k_0)\,\varepsilon}$$

For this integral to converge, the <u>exponential</u> dependence should not dominate $1/k^2$. Hence positive values of x_0 ($x_0 > 0$) mean that we need positive imaginary component of k^0.

But, if that is true, then the integral will be evaluated when the contour wraps the two poles above the plane.



This will give a non-zero result. Note that for $x_0 < 0$, contour would have been closed in the lower plane giving zero !!

2) $Z(J) = \int \mathcal{D}\phi \, e^{-S + \int J(x)\varphi(x) d^d x}$

$$\Rightarrow \quad \frac{1}{Z[0]} \frac{\delta Z[J]}{\delta J(x_1)}\bigg|_{J=0} = \underbrace{\frac{1}{Z[0]} \int \mathcal{D}\varphi \, \varphi(x_1) e^{-S[\phi]}}_{\langle \phi(x_1) \rangle}$$

Similarly,

$$\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \frac{1}{Z[0]} \int \mathcal{D}\varphi \, \varphi(x_1)\varphi(x_2) e^{-S[\varphi]} \quad - ①$$

$$= \langle \phi(x_1)\varphi(x_2) \rangle$$

we can write using the concept of time-ordering as

$$\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \langle \varphi(x_1)\varphi(x_2)\rangle = \langle 0 | T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)] | 0 \rangle \quad - ②$$

Eq. ② basically means that functional derivatives of $Z[J]$ give vacuum expectation values of time-ordered product of field operators.

Note:

<u>Euclidean</u> <u>Propagator</u> $\left(e^{-S/\hbar} \overset{\hbar=1}{\approx} e^{-S[\phi]} \right)$

$$Z_0[J] = \int \mathcal{D}\varphi \, e^{-\int [\frac{1}{2}(\partial\varphi)^2 + \frac{m^2\varphi^2}{2}] d^d x + \int J(x)\varphi(x) d^d x}$$

$$\tilde{\varphi}(p) = \int d^d x \; e^{-ip.x} \, \varphi(x)$$

$$\varphi(x) = \int \frac{d^d p}{(2\pi)^d} \, e^{ip.x} \, \tilde{\varphi}(p)$$

Term in the exponential is then:

$$-\int \frac{d^d p}{(2\pi)^d} \left[\frac{1}{2} \tilde{\phi}(p)(p^2+m^2)\tilde{\varphi}(-p) - \tilde{J}(p)\tilde{\varphi}(-p) \right]$$

Complete the square of the term in square brackets:

$$\frac{1}{2}\left[\tilde{\varphi}(p) - \frac{1}{p^2+m^2}\tilde{J}(p)\right](p^2+m^2)\left[\tilde{\varphi}(-p) - \frac{1}{p^2+m^2}\tilde{J}(-p)\right]$$

$$\underbrace{-\frac{1}{2}\tilde{J}(p)\frac{1}{p^2+m^2}\tilde{J}(-p)}$$

subtracted this to fix everything.

Now let

$$\tilde{\varphi}(p) = \tilde{\phi}(p)' + (p^2+m^2)^{-1}\tilde{J}(p)$$

$$Z_0[J] = \int \mathcal{D}\tilde{\varphi}' \, e^{-\frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p)'(p^2+m^2)\tilde{\phi}(-p)' + \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

$$= Z_0[0] \, e^{\frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \tilde{J}(p)(p^2+m^2)^{-1}\tilde{J}(-p)}$$

↓ x-space

$$Z_0[J] = Z_0[0] \, e^{\frac{1}{2}\int d^d x' \int d^d x'' J(x')\Delta(x'-x'')J(x'')}$$

gives $\Delta(x'-x'') = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip.(x'-x'')}}{p^2+m^2}$.

So, the Euclidean propagator is:

$$\mathcal{D}_E(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik.(x-y)}}{K^2+m^2}$$

no '$i\varepsilon$' prescription required.

Space average of $\mathcal{D}_E(x-y)$ means: roughly;

$$\underbrace{\int}_{\text{'space average'}} \mathcal{D}_E(x-y) = \underline{\underline{\mathcal{D}_E(\tau-\tau')}}$$

four vector

Euclidean time.

required.

$$\bar{\mathcal{D}}_E(\tau-\tau') = \int \frac{dk^0}{(2\pi)} \frac{e^{-ik^0(\tau-\tau')}}{\omega^2+m^2} \quad ; k^0 := \omega$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(\tau'-\tau)}}{\omega^2+m^2}$$

So the integral can be evaluated for $\tau' > \tau$ by closing in the upper half plane & $\tau' < \tau$ by closing in the lower half plane

Residues at $\omega = \pm im$

Digression:-

$$f(x,t) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{f}(k,t)$$

Space average is:

$$\bar{f}(t) = \int d^3x \; f(\vec{x},t)$$

$$= \int d^3x \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{f}(k,t)$$

$$= \frac{1}{(2\pi)^3} (2\pi)^3 \delta^3(\vec{k}) \tilde{f}(k,t)$$

$$= \tilde{f}(0,t)$$

Space average of $\mathcal{D}_E(x-y)$ just amounts to doing the dk^0 integration when space-components are set to zero.

$$\bar{D}_E(\tau-\tau') = \underbrace{\frac{\theta(\tau-\tau')\, e^{\omega(\tau'-\tau)}}{2\omega}}_{\text{lower half}} + \underbrace{\frac{\theta(\tau'-\tau)\, e^{\omega(\tau-\tau')}}{2\omega}}_{\text{upper half}}$$

$$= \frac{e^{-\omega|\tau-\tau'|}}{2\omega}$$

(No 'i' dependence on exponent since $\omega = \pm im$ takes that out).

$$\bar{D}_E(\tau-\tau') = \frac{e^{-\omega|\tau-\tau'|}}{2\omega}$$

function of Euclidean <u>time</u>.

Now, if we substitute $\tau = it$ and $\tau' = it'$ we get,

$$D_{Minkowski}(t-t') = \frac{e^{-i\omega|t-t'|}}{2\omega}$$

3. $Z = e^{\frac{1}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)}$

and $E = \frac{-1}{\beta} \ln Z$

$$= \frac{-1}{2\beta} \int d^4x\, d^4y\, J(x)\, D(x-y)\, J(y) \quad - \text{①}$$

Now, it makes more sense to pass over to k-space and solve this. Writing $D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2-m^2+i\varepsilon}$

and using in ①, we get :

$$E = \frac{-1}{2\beta} \underbrace{\int d^4x\, J(x)\, e^{ikx}} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-iky}}{k^2-m^2+i\varepsilon} \int d^4y\, J(y)$$

But note that $\underbrace{\quad}$ is the fourier transform of $J(x)$ i.e $\tilde{J}(k)$ and similarly for $J(y)$.

We then have ;

$$E = \frac{-1}{2\beta} \int d^4k \frac{\tilde{J}(k)\,\tilde{J}(-k)}{k^2-m^2+i\varepsilon}$$

$$= \frac{-1}{2\beta} \int d^4k \frac{|\tilde{J}(k)|^2}{k^2-m^2+i\varepsilon} \quad - \text{②}$$

Now, we use the $J(x)$ given in the problem.

$$J(x) = \theta(T-t)\,\theta(T+t)\left\{a_1\,\delta(x) + a_2\,\delta(x-R)\right\}$$

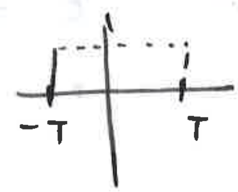
Let's take fourier transform of $J(x)$

$$\tilde{J}(k) = \int \frac{d^4x}{(2\pi)^4}\, J(x)\, e^{ik\cdot x}$$

$$= \int_{-T}^{T} \frac{dt\, e^{ik^0x_0}}{(2\pi)} \int \frac{d^3x}{(2\pi)^3}\left\{a_1\,\delta(x) + a_2\,\delta(x-R)\right\} e^{-i\vec{k}\cdot x}$$

$\theta(T-t) = 1$ when $T-t>0$, $\boxed{T>t}$

$\theta(T+t) = 1$ if $T+t>0$, $T>-t$, $\boxed{-T<t}$



Let us define $k^0 := \omega$

$$= \left|\frac{e^{i\omega t}}{i\omega}\right|_{-T}^{T} \int \frac{d^3x}{(2\pi)^3}\left\{a_1\delta(x) + a_2\delta(x-R)\right\} e^{-i\vec{k}\cdot x}$$

$$= \frac{2\sin\omega T}{2\omega\pi}\left[\int \frac{d^3x}{(2\pi)^3}\, a_1\delta(x)e^{-i\vec{k}\cdot x} + \int \frac{d^3x}{(2\pi)^3}\, a_2\,\delta(x_{-R})\,e^{-i\vec{k}\cdot x}\right]$$

$$= \frac{\sin\omega T}{\omega\pi}\left[a_1 + a_2\, e^{-i\vec{k}\cdot\vec{R}}\right]$$

$$\tilde{J}(k)\,\tilde{J}(-k) = |\tilde{J}(k)|^2 = \frac{\sin^2\omega T}{\omega^2\pi^2}\left[a_1^2 + a_2^2 + 2a_1a_2\cos\vec{k}\cdot\vec{R}\right]$$

Plugging (3) to (2) we get,

$$E = \frac{-1}{2\beta} \int d^4k \ \frac{\sin^2 \omega T}{\omega^2 \pi^2} \left[a_1^2 + a_2^2 + 2a_1a_2 \cos k \cdot \bar{R} \right] \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$= \frac{-1}{2\beta} \int d\omega \ \frac{\sin^2 \omega T}{\omega^2 \pi^2} \int d^3k \left[\frac{a_1^2 + a_2^2 + 2a_1a_2 \cos \vec{k} \cdot \vec{R}}{k^2 - m^2 + i\varepsilon} \right] \quad \cdots (4)$$

ω^2 still here!

But we know

$$\int d\omega \ \frac{\sin^2(\omega T)}{\omega^2} = \pi T \quad \cdots (5)$$

[standard integral]

using (5) in (4) we get,

$$E = \frac{-1}{2\beta} \frac{T}{\pi} \int d^3k \ \frac{a_1^2 + a_2^2 + 2a_1a_2 \cos \vec{k} \cdot \vec{R}}{k^2 - m^2 + i\varepsilon}$$

$$= \underbrace{\frac{-1}{2\beta} \frac{T}{\pi} \int d^3k \ \frac{a_1^2 + a_2^2}{k^2 - m^2 + i\varepsilon}}_{I_1} \ \underbrace{- \frac{2a_1a_2}{2\beta} \frac{T}{\pi} \int \frac{d^3k \cos \vec{k} \cdot \vec{R}}{k^2 - m^2 + i\varepsilon}}_{I_2}$$

We can do I_1 but it does not make any sense since it has no R dependence and it will anyways fall out in calculation of force..

$$F = \frac{-\partial E}{\partial R} \cong \frac{-\partial}{\partial R}\left[\frac{-a_1 a_2 T}{2\beta\pi}\int d^3k \, \frac{e^{i\vec{k}\cdot\vec{R}} + e^{-i\vec{k}\cdot\vec{R}}}{k^2 - m^2 + i\varepsilon}\right]$$

Using Integral table, we evaluate and find in the $\underline{\text{lt } \epsilon \to 0}$.

Note $T \gg \frac{1}{m}$

$\Rightarrow m \gg \omega$

$\underline{(k^0)^2 \ll (m)^2}$.

$$\cong \frac{-\partial}{\partial R}\left[\frac{-a_1 a_2 T}{2\beta\pi}\,\frac{e^{-m|R|}}{2\pi R}\right]$$

$$\cong \frac{a_1 a_2 T}{2\pi\beta}\left[\frac{-m e^{-m|R|}}{2\pi R} - \frac{e^{-m|R|}}{4\pi R^2}\right]$$

$$\cong \frac{-a_1 a_2 T}{2\pi\beta}\left[\frac{m e^{-m|R|}}{2\pi R} + \frac{e^{-m|R|}}{4\pi R^2}\right]$$

$\cong -$ ($\oplus$ve quantity !!)

The force dies out with increasing $|\bar{R}|$ ..

$$R \gg \frac{1}{m} \Rightarrow m R \gg 1$$

$$-m|R| \lll 1$$

$\hookrightarrow$ dies out sufficiently quickly..

Attractive since it has a negative sign in front !!

Note :- Later realized that the form given for '$\underline{\underline{Z}}$' does not have a factor of "$\underline{i}$".

This clearly hints that we should work in the Euclidean space where propagator has factor $\frac{1}{k^2+m^2}$ ..

I did it using Minkowski propagator & wick rotated back later.

✓

10th Feb 2015

1) $$\phi_n^c = \frac{\delta W(J)}{\delta J_n} = \langle \phi \rangle_J$$

if $$\Gamma(\varphi_c) = W(J) - \sum_n J_n \phi_n^c$$

a) Show that

$$\frac{\delta \Gamma}{\delta \phi_m^c} = -J_m$$

Sol$^{\underline{n}}$:-
$$\frac{\delta \Gamma}{\delta \phi_m^c} = \frac{\delta W(J)}{\delta \phi_m^c} - \sum_n \frac{\delta}{\delta \phi_m^c}\left(J_n \phi_n^c\right)$$

$$= \frac{\delta W(J)}{\delta J_n}\frac{\delta J_n}{\delta \phi_m^c} - \sum_n J_n \frac{\delta \phi_n^c}{\delta \phi_m}$$

$$= \phi_n^c \cancelto{0}{\frac{\delta J_n}{\delta \phi_m^c}} - \sum_n J_n \delta_{nm}$$

$$= -J_m \checkmark$$

b) Verify

$$\sum_n \frac{\delta^2 W(J)}{\delta J_1 \delta J_n} \frac{\delta^2 \Gamma}{\delta \phi_n^c \delta \phi_2^c} = -\delta_{12}$$

Sol^n: $\sum_n \frac{\delta}{\delta J_1} \frac{\delta W(J)}{\delta J_n} \frac{\delta}{\delta \phi_2^c} \frac{\delta \Gamma}{\delta \phi_n^c}$

$$\sum_n \frac{\delta}{\delta J_1} \phi_n^c \frac{\delta}{\delta \phi_2^c}(-J_n)$$

$$-\sum_n \frac{\delta \phi_n^c}{\delta \phi_2^c} \frac{\delta J_n}{\delta J_1}$$

$$-\sum_n \delta_{2,n} \delta_{n,1}$$

$$-\delta_{12} \quad ✓$$

c) Using the result

$$\frac{\partial}{\partial \alpha} M^{-1}(\alpha) = -M^{-1} \frac{\partial M}{\partial \alpha} M^{-1} \quad \cdots \text{(A)}$$

where $M_{\alpha\beta} = \frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta \phi_\beta}$ and $M_{\alpha\beta}^{-1} = \frac{-\delta^2 W}{\delta J_\alpha \delta J_\beta}$

Sol^n: Using the eq^n (A) given;

$$\frac{\partial M_{12}^{-1}}{\partial J_3} = -M_{1\alpha}^{-1} \frac{\partial M_{\alpha\beta}}{\partial J_3} M_{2\beta}^{-1}$$

$$\frac{\partial}{\partial J_3} M_{12}^{-1} = \sum_{\alpha,\beta,\xi} - M_{1\alpha}^{-1} \frac{\partial M_{\alpha\beta}}{\partial \phi_\xi} \frac{\partial \phi_\xi}{\partial J_3} M_{2\beta}^{-1} \quad \cdots \quad (B)$$

Now $\displaystyle M_{12}^{-1} = \frac{-\delta^2 W}{\delta J_1 \delta J_2}$

$$M_{1\alpha}^{-1} = \frac{-\delta^2 W}{\delta J_1 \delta J_\alpha}$$

$$M_{2\beta}^{-1} = \frac{-\delta^2 W}{\delta J_2 \delta J_\beta}$$

Using above three eqn's in (B) we get,

$$\frac{-\partial}{\partial J_3}\left(\frac{\delta^2 W}{\delta J_1 \delta J_2}\right) = \sum_{\alpha,\beta,\xi} - \frac{\delta^2 W}{\delta J_1 \delta J_\alpha} \underline{\underline{\frac{\delta M_{\alpha\beta}}{\delta \phi_\xi}}} \, \underline{\underline{\frac{\delta \phi_\xi}{\delta J_3}}} \frac{\delta^2 W}{\delta J_2 \delta J_\beta} \quad - \quad (C)$$

Let's calculate underlined terms

$$\frac{\delta M_{\alpha\beta}}{\delta \phi_\xi} = \frac{\delta}{\delta \phi_\xi} \frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta \phi_\beta}$$

$$= \Gamma_{\alpha\beta\xi} \quad - \quad (D)$$

and $\displaystyle \frac{\delta \phi_\xi}{\delta J_3} = \frac{\delta}{\delta J_3} \frac{\delta W}{\delta J_\xi}$

$$= \frac{\delta^2 W}{\delta J_3 \delta J_\xi} = -G_{\xi 3} \quad - \quad (E)$$

using D & E in C we get,

$$-\frac{\delta^2 W(J)}{\delta J_1 \delta J_2 \delta J_3} = \sum_{\alpha,\beta,\xi} + \frac{\delta^2 W}{\delta J_1 \delta J_\alpha} \Gamma_{\alpha\beta\xi} \, G^c_{\xi 3} \frac{\delta^2 W}{\delta J_\beta \delta J_2}$$

Change $\xi \to \gamma$, dummy.

$$\Rightarrow G_C^3(1,2,3) = \sum_{\alpha,\beta,\gamma} G_c^{(2)}(1,\alpha) G_C^{(2)}(2,\beta) G_c^{(2)}(3,\gamma) \Gamma^{(3)}_{\alpha\beta\gamma}$$

2) Dirac Equation

$(i\not{\partial} + m)\psi = 0$

In momentum space $(\not{p} + m)\psi(p) = 0$

Rest frame of particle $p = (m, 0, 0, 0)$; since it is Lorentz invariant

$$\not{p} = \gamma^\mu p_\mu$$
$$= \gamma^0 p_0 + \cdots$$
$$= \gamma^0 m$$

This means ; $(\gamma^0 + \mathbb{I})\psi(p) = 0$

using $\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$; we get

$\begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}\psi(p) = 0$ (excluding factors of 2 in coefficient)

$\downarrow$

This clearly has been reduced from 4×4 to 2×2 non-zero matrix $(\mathbb{I}_2)$; hence 2 d.o.f.

Two of the four components of ψ are zero.

Also, note that $(\gamma^0 + \mathbb{I})^2$ is upto some factor equal to $(\gamma^0 + \mathbb{I})$.

i.e $\quad A^2 = A \quad$; Projection operator

Hence, it projects original 4 component spinor to a subspace where it is only 2 component hence in agreement with Wigner.

2) <u>Rarita Schwinger</u> Spin $3/2$

$$\mathcal{L} = \bar{\Psi}_\mu \gamma^{\mu\nu\lambda} \partial_\nu \Psi_\lambda + m \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu$$

E.O.M is given by $\left(\begin{array}{c} \text{E-L w.r.t } \bar{\Psi} \\ \text{Euler-Lagrange equation} \end{array} \right)$

$$\gamma^{\mu\nu\lambda} \partial_\nu \Psi_\lambda + m \gamma^{\mu\nu} \Psi_\nu = 0$$

(Relabelling of indices)..

$$\Rightarrow \quad \gamma^{\mu\lambda\nu} \partial_\lambda \Psi_\nu - m \gamma^{\mu\nu} \Psi_\nu = 0$$

$$\Rightarrow \left(\gamma^{\mu\lambda\nu} \partial_\lambda - m \gamma^{\mu\nu} \right) \Psi_\nu = 0$$

✓

b) Act on E.O.M with ∂_μ

$$\underbrace{\partial_\mu \left(\gamma^{\mu\lambda\nu} \partial_\lambda \right.}_{} - \left. m \gamma^{\mu\nu} \right) \Psi_\nu = 0$$

Goes to zero ($\partial_\mu \partial_\lambda$ is symmetric under change of indices and $\gamma^{\mu\lambda\nu}$ is antisymmetric)

$$\Rightarrow \quad \partial_\mu \gamma^{\mu\nu} \Psi_\nu = 0 \qquad \cdots \quad (1)$$

Write $\gamma^{\mu\nu} = \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} - \gamma^\nu \gamma^\mu$

$$= \frac{1}{\cancel{2}} \cancel{2} \eta^{\mu\nu} \mathbb{I} - \gamma^{\nu}\gamma^{\mu}$$

$$= \eta^{\mu\nu} - \gamma^{\nu}\gamma^{\mu} \quad \cdots \quad ②$$

Now using ②, ① becomes

$$\partial_\mu \left(\eta^{\mu\nu} \psi_\nu - \gamma^\mu \gamma^\nu \psi_\nu \right) = 0$$

$$\Rightarrow \quad \partial_\mu \psi^\mu - (\gamma^\mu \partial_\mu)(\gamma^\nu \psi_\nu) = 0$$

$$\Rightarrow \quad \partial_\mu \psi^\mu = (\gamma^\mu \partial_\mu)(\gamma^\nu \psi_\nu) \checkmark .$$

c) Act on E.o.M with γ_μ .

Few identities to be used:

* $\gamma^{\mu\nu\lambda} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda]} = \frac{1}{3}\left(\gamma^\mu\gamma^{\nu\lambda} + \gamma^\nu\gamma^{\lambda\mu} + \gamma^\lambda\gamma^{\mu\nu}\right)$

$\longrightarrow$ cyclic permutation of indices.. μ, ν, λ

* $\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda = 2\eta^{\nu\lambda}$
* $\gamma^\mu\gamma_\mu = 4\mathbb{I}_4$
* $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$

E.O.M reads :

$$\left(\gamma^{\mu\lambda\nu}\partial_\lambda - m\gamma^{\mu\nu}\right)\psi_\nu = 0$$

Act with γ_μ

$$\gamma_\mu\left(\gamma^{\mu\lambda\nu}\partial_\lambda - m\gamma^{\mu\nu}\right)\psi_\nu = 0$$

$$\gamma_\mu\left\{\frac{1}{3}\left(\gamma^\mu\gamma^{\lambda\nu} + \gamma^\lambda\gamma^{\nu\mu} + \gamma^\nu\gamma^{\mu\lambda}\right)\partial_\lambda\psi_\nu\right\} - m\gamma_\mu\gamma^{\mu\nu}\psi_\nu = 0$$

From here onward, factors are not taken care of since we have to just prove $\boxed{\gamma^\mu\partial_\mu = 0}$

$$\left(\gamma_\mu\gamma^\mu\gamma^{\lambda\nu} + \gamma_\mu\gamma^\lambda\gamma^{\nu\mu} + \gamma_\mu\gamma^\nu\gamma^{\mu\lambda}\right)\partial_\lambda\psi_\nu - m\gamma_\mu\left(\frac{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu}{2}\right)\psi_\nu = 0$$

$$\left[4\gamma^{\lambda\nu} + \gamma_\mu\gamma^\lambda\left(\frac{1}{2}\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) + \gamma_\mu\gamma^\nu\left(\frac{1}{2}\gamma^\mu\gamma^\lambda - \gamma^\lambda\gamma^\mu\right)\right]\partial_\lambda\psi_\nu - \frac{m\gamma_\mu}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\psi_\nu = 0$$

Simplifying

$$- \frac{m}{2}\left(4\gamma^\nu + 2\gamma^\nu\right)\psi_\nu = 0$$

$$\left[4\gamma^{\lambda\nu} + \cancel{\gamma^\nu\gamma^\lambda} + \cancel{2\eta^{\nu\lambda}} - \cancel{\gamma^\nu\gamma^\lambda} - \cancel{2\eta^{\nu\lambda}}\right]\partial_\lambda\psi_\nu - 3m\gamma^\nu\psi_\nu = 0$$

Dropping all factors for convenience,

$$\gamma^{\lambda\nu}\partial_\lambda\psi_\nu = \gamma^\nu\psi_\nu$$

Now the L.H.S $= 0$; from Eq. 1 on Page - 7

$$\Rightarrow \quad \gamma^{\mu}\psi_{\mu} = 0 \qquad (\text{relabell } \mu \mapsto \nu)$$
since dummy.

d) Constraint 1 : $\partial_{\mu}\psi^{\mu} = (\gamma^{\mu}\partial_{\mu})(\gamma^{\nu}\psi_{\nu}) \quad \ldots (1)$

Constraint 2 : $\gamma^{\mu}\psi_{\mu} = 0 \quad \ldots (2)$

The $\gamma^{\mu}\psi_{\mu} = 0$ constraint is like a gauge; infact harmonic gauge. Interesting to note is that (2) in (1) implies $\boxed{\partial_{\mu}\psi^{\mu} = 0}$

So, now we have

$\boxed{\partial_{\mu}\psi^{\mu} = 0}$ and $\boxed{\gamma^{\mu}\psi_{\mu} = 0}$

Note : we have suppressed the spinor index 'α' on ψ_{μ} all throughout.

Initially ψ_{μ}^{α} had 16 d.o.f.

$\downarrow \; \partial_{\mu}\psi^{\mu} = 0$

reduced it to 12

and then the second constraint reduced it further to 8 $(16 \rightarrow 12 \rightarrow 8)$..

But, we need to do more..

e) Including the constraints in the E.O.M derived in part (a) gives :

$$-\gamma^{\mu\lambda\nu}\partial_\lambda \psi_\nu + m\gamma^{\mu\nu}\psi_\nu = 0 \qquad \leftarrow \text{E.O.M}$$

Impose $\partial.\psi = 0$ i.e $\partial_\mu \psi^\mu = 0$

and $\gamma^\mu \psi_\mu = 0$

$$m\gamma^{\mu\nu}\psi_\nu \longrightarrow \frac{m}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\psi_\nu$$

$$= \frac{m}{2}\left(2\gamma^\mu\gamma^\nu - 2\eta^{\mu\nu}\right)\psi_\nu$$

$$= m\left(\gamma^\mu\gamma^\nu\psi_\nu - \psi^\mu\right)$$

$$= m\left(\gamma^\mu(\gamma.\psi) - \psi^\mu\right)$$

But $\gamma.\psi = 0$

So $m\gamma^{\mu\nu}\psi_\nu \longmapsto -m\psi^\mu$ ✓.

The $-\gamma^{\mu\lambda\nu}\partial_\lambda \psi_\nu$ part gives (leaving out factors like $\frac{1}{3}$ etc.

$$-\left[\gamma^\mu\gamma^{\lambda\nu} + \gamma^\lambda\gamma^{\nu\mu} + \gamma^\nu\gamma^{\mu\lambda}\right]\partial_\lambda\psi_\nu$$

$$-\left\{\gamma^\mu(\gamma^\lambda\gamma^\nu - \eta^{\lambda\nu}) + \gamma^\lambda(\gamma^\nu\gamma^\mu - \eta^{\nu\mu}) + \gamma^\nu(\gamma^\mu\gamma^\lambda - \eta^{\mu\lambda})\right\}\partial_\lambda\psi_\nu$$

$$-\Big[\gamma^\mu(\gamma\cdot\partial)\cancel{(\gamma\cdot\psi)} - \cancel{\partial^\nu\psi_\nu} + \gamma^\lambda\cancel{(\gamma\cdot\psi)}(\gamma\cdot\partial)$$
$$- \gamma^\lambda\partial_\lambda\psi^\mu$$
$$+ \gamma^\nu\cancel{(\gamma\cdot\psi)}(\gamma\cdot\partial) - \cancel{\gamma^\nu\partial^\mu\psi_\nu}$$

we get

$$-\gamma^{\mu\lambda\nu}\partial_\lambda\psi_\nu \longmapsto + \gamma^\lambda\partial_\lambda\psi^\mu$$
$$+ \not{\partial}\psi^\mu$$

So, constrained E.O.M is

$$(\not{\partial} - m)\psi^\mu = 0$$

Note that spinor index on ψ^μ is still suppressed.

We can remove the last four d.o.f following the method we did for Dirac eq$^{\underline{n}}$.

Taking Fourier transform of $(\not{\partial} - m)\psi^\mu = 0$

gives $(i\not{p} - m)\psi^\mu = 0$

going to a frame where $p^\mu = (m, 0, 0, 0)$

gives

$$(i\gamma^0 - \mathbb{I})\psi^\mu = 0 \quad *$$ (there should not be an '$\underline{\underline{i}}$' here)..

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

! Following same prescription as discussed in Dirac case we can show that

$$A = i\gamma^0 - \mathbb{I} = A^2$$

(Projection operator)

and it reduces the d.o.f from $\underline{8 \text{ to } 4}$..

<u>Note</u> :- The factor of '$\underline{\underline{i}}$' survives since $\mathcal{L}$ <u>had</u> no '$\underline{\underline{i}}$' but, don't know the interpretation..

We remain with $\underline{8 - 4 = 4}$ d.o.f which is the correct for spin $3/2$ i.e $\underline{2\left(\frac{3}{2}\right) + 1 = 4}$ ✓.