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RAGHAV..

1) $$\frac{d\alpha}{d\ln\mu} = -2\alpha\left(\left(\frac{11C_A - 2N_f}{3}\right)\frac{\alpha}{4\pi} + b_2\left(\frac{\alpha}{4\pi}\right)^2 + \cdots\right) \quad - (1)$$

where $\alpha = \dfrac{g^2}{(4\pi)^2}$

$$\left.\begin{array}{l} C_A = 3 \\ C_f = \frac{4}{3} \end{array}\right\} \text{for } SU(3)$$

Expression for $\alpha(\mu)$ ...

Writing (1) for the one-loop ;

$$\mu\frac{d\alpha}{d\mu} = -2\alpha\left(\left(\frac{11C_A - 2N_f}{3}\right)\frac{\alpha}{4\pi}\right)$$

$$\mu\frac{d\alpha}{d\mu} = \frac{-\alpha^2}{2\pi}\left(\underbrace{\left(\frac{11C_A - 2N_f}{3}\right)}_{\chi}\right)$$

$$\Rightarrow \int_{\alpha_0}^{\alpha} \frac{-d\alpha}{\alpha^2\chi} = \frac{1}{2\pi}\int_{\mu_0}^{\mu}\frac{d\mu}{\mu}$$

$$\frac{1}{\chi}\left|\frac{1}{\alpha}\right|_{\alpha_0}^{\alpha} = \frac{1}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{\chi}\left(\frac{1}{\alpha} - \frac{1}{\alpha_0}\right) = \frac{1}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)$$

$$\frac{1}{\alpha} = \frac{1}{\alpha_0} + \frac{\chi}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)$$

$$\cancel{\alpha_0}\alpha \Rightarrow \alpha_0 = \alpha + \frac{\chi\alpha\alpha_0}{2\pi}\ln\left(\frac{\mu}{\mu_0}\right)$$

$$\Rightarrow \alpha(\mu) = \frac{\alpha_0}{1 + \alpha_0\chi\ln(\mu/\mu_0)}$$

10) for $N = 2$

$C_A = 2$

$C_f = 3/4$

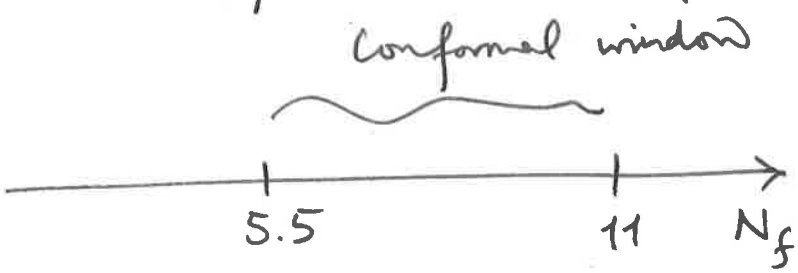
β- function develops a zero when the signs of b_0 & b_1 are opposite. In this regime, the asymptotic freedom is lost.

$$34\, C_A^2 < 2(5C_A + 3C_f)N_f$$

$$136 < \frac{98}{4} N_F$$

$$\boxed{n_F > 5.55}$$

So, when $11 > n_F > 5.55$, β- function can develop a zero away from origin.



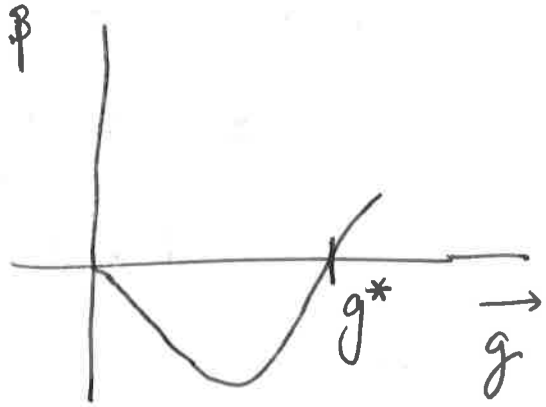
Sketch β-f^n for $N_f = 0$ & $N_f > 11$

for $\underline{N_f = 0}$

$$b_1 = \frac{11\, C_A}{3} = \frac{22}{3} \quad ; \quad b_2 = \left(\frac{34}{3}\right)4 = \frac{136}{3}$$

It is interesting to note that there are no zeros of β - fn away from origin in either of them since that happens for $5.5 < N_f < 11$

For ex: for $N_f = 8$



c) $$\beta(\alpha) = \frac{-b_0\alpha^2}{4\pi} + \frac{b_1\alpha^3}{16\pi^2}$$

$$\beta(\alpha^*) = 0$$

$$\Rightarrow \frac{b_0\alpha^2}{\cancel{4\pi}} = \frac{b_1\cancel{\alpha^3}\alpha}{\cancel{16\pi^2}4\pi}$$

$$\frac{b_0}{b_1} = \frac{\alpha}{4\pi}$$

$$\frac{(11C_A - 2N_f)}{34C_A^2 - 10C_An_F - 6C_Fn_F} = \frac{\alpha}{4\pi}$$

$$\frac{22 - 2n_F}{156 - 20n_F - \frac{18}{4}n_F} = \frac{\alpha}{4\pi}$$

From part (a) we see that if $\mu \to 0$

$$\alpha\Big|_{\mu=0} \longrightarrow \text{some constant value.}$$

and β - fn is $\underline{\underline{\text{zero}}}$..

This is the case of IR fixed point. The theory behaves in a quasi-conformal way. Since, we don't have any notion of mass scale (conformal). This IR fixed point is also known as Banks-Zaks fixed point.

$$Q^2 A_\mu^a = \mathcal{D}_\mu^{ab}(Qc^b) - \frac{1}{2} g f^{abc}(\partial_\mu c^c c^b) - \frac{1}{2} g^2 f^{bdc} f^{cae} A_\mu^e c^d c^b \quad \cdots (3)$$

Note that since f^{abc} is anti-symmetric, we can recast $(\partial_\mu c^{[c})c^{b]}$ as

$$(\partial_\mu c^c)c^b = \frac{1}{2} \partial_\mu (c^c c^b) \quad ✓ \quad \cdots (4)$$

Using $Qc^b = -\frac{1}{2} g f^{bca} c^c c^a$ in (3) gives

$(* f^{abc} = -f^{bac})$

$$Q^2 A_\mu^a = \frac{1}{2} g \cancel{f^{abc}} (\partial_\mu c^c c^b) - \frac{1}{2} g \cancel{f^{abc}} (\partial_\mu c^c c^b) + \underline{\frac{1}{2} g^2 f^{bde} \cancel{f^{cae}} A_\mu^e c^d c^b} - \frac{1}{2} g^2 f^{bde} \cancel{f^{cae}} A_\mu^e c^d c^b$$

(Note: relabelling of indices in the underlined term)

$$= 0 \quad ✓$$

3) $\gamma_A = \frac{1}{2} \frac{\partial \ln Z_3}{\partial \ln \mu}$ ... ①

where Z_3 in $\overline{MS}$ scheme is given by:

$$Z_3 = 1 + \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4)$$

Rewrite ① as

$$\gamma_A = \frac{1}{2} \frac{\partial \ln Z_3}{\partial g} \frac{\partial g}{\partial \ln \mu} \quad \dots \quad ②$$

We derived the expression for $\frac{\partial g}{\partial \ln \mu}$ to $O(\varepsilon)$ as

$$\frac{\partial g}{\partial \ln \mu} = \frac{-\varepsilon g}{2}$$

$$\frac{\partial \ln Z_3}{\partial g} = \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] \frac{2g}{8\pi^2 \varepsilon} + O(g^4)$$

$\ln(1+x) \sim x$ to $O(g^2)$

Then, Eq. ② becomes

$$\gamma_A = \frac{1}{2} \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] \frac{\cancel{2} g}{8\pi^2 \cancel{\varepsilon}} \frac{-\cancel{\varepsilon} g}{\cancel{2}}$$

$$= \frac{-g^2}{16\pi^2} \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R)\right] + O(g^4)$$