k solution discussed in class :

$$\varphi = v \tanh\left(\frac{m}{2}(x - x_0)\right)$$

raghav
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Lorentz boost solution (i.e moving soliton) is :

$$\varphi(x,t) = v \tanh\left[\frac{\gamma m}{2}(x - x_0 - \beta t)\right] \quad \cdots (1)$$

E^n of motion reads :

$$\Box\varphi(x,t) + V'(\varphi(x,t)) = 0 \quad \cdots (2)$$

where $\Box = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$

Signature used in problem $(+ - - -)$

$$\Box\varphi(x,t) = \frac{\beta^2}{(1-\beta^2)}\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right) - \frac{1}{(1-\beta^2)}\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)$$

$$= \left(\frac{\beta^2 - 1}{1-\beta^2}\right)\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)$$

$$= -\varphi''\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right) \quad \cdots (3)$$

Note that I've set $x_0 \to 0$ & $\frac{m}{2} = 1$ to keep argument simple. They won't affect what we set to prove through the entire problem.

Now we note that

$$\Rightarrow \frac{1}{2}\left(\frac{1+\beta^2}{1-\beta^2}\right) + \frac{1}{2}$$

$$= \frac{1}{1-\beta^2}$$

Hence,

$$\underbrace{T(x,t)}_{\text{En. density}} = \frac{1}{1-\beta^2}\left(\varphi'\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)\right)^2$$

$$\text{Energy of moving soliton} = \int_{-\infty}^{\infty} T(x,t)\,dx$$

$$= \frac{1}{1-\beta^2}\int_{-\infty}^{\infty} \varphi'\left(\frac{x-\beta t}{\sqrt{1-\beta^2}}\right)^2 dx$$

Now let $\frac{x-\beta t}{\sqrt{1-\beta^2}} = \xi$

$$dx = \sqrt{1-\beta^2}\,d\xi$$

$$= \frac{1}{\sqrt{1-\beta^2}}\underbrace{\int_{-\infty}^{\infty} \varphi'(\xi)^2\,d\xi}_{\text{Rest Energy}}$$

$$= \gamma E_{\text{rest energy}}$$

$$= \gamma M$$

2. Let $\Gamma(g) = \int d^3x \ \varepsilon^{ijk} \ \mathrm{Tr}\left\{ (g\partial_i g^+)(g\partial_j g^+)(g\partial_k g^+) \right\}$

$$= \int d^3x \ \mathrm{Tr}\,(g\partial g^+)^3 \quad \cdots \quad ①$$

We've to show that ① is invariant under smooth deformations.

Sol$^{\underline{n}}$: As a digression, let's prove a result useful later on.

$$\Gamma(g_1 g_2) = \Gamma(g_1) + \Gamma(g_2) \quad \cdots \quad ②$$

Proof:- $(g_1 g_2)\partial(g_1 g_2)^\dagger = g_1 g_2 \partial g_2^+ g_1^+ + g_1 g_2 g_2^+ \partial g_1^+$

$$= g_1 (g_2 \partial g_2^+ + \partial g_1^+ g_1) g_1^+$$

We used the fact $gg^+ = \mathbb{1}$

Infact, $g_1 g_1^+ = g_2 g_2^+ = \mathbb{1}$

Let $g_2 \partial g_2^+ = X$

and, $\partial g_1^+ g_1 = Y$

$$\Gamma(g_1 g_2) = \int \mathrm{Tr}\,(X+Y)(X+Y)(X+Y)\, d^3x$$

Now, we can show that $\Gamma(g)$ is invariant under $g \to g + \delta g$.

$$g \to g + \delta g$$
$$\to g\left(1 + \frac{\delta g}{g}\right) \quad \dots \quad 4$$

$$\Gamma(g+\delta g) = \Gamma\left(g\left(1+\frac{\delta g}{g}\right)\right) = \Gamma(g) + \Gamma\left(1+\frac{\delta g}{g}\right)$$

$$= \Gamma(g) + \underbrace{\Gamma\left(1+\frac{\delta g}{g}\right)}_{\text{zero}}$$

But $\delta g \to 0$ at $x = \pm\infty$ and $\Gamma(1) = 0$

$$\Rightarrow \quad \Gamma(g+\delta g) = \Gamma(g) \; \checkmark.$$

Co-ordinate transformation independent:

Since $\Gamma(g)$ is invariant under $g + \delta g$, we can argue that it will also be invariant under $x + \delta x$ since we can think of 'δg' deformation as inducing some change in coordinate (since $g(x)$). ✓.

2) Let's write down the variation of $\delta(g\partial_i g^\dagger)$

$$\delta(g\partial_i g^\dagger) = \delta g\, \partial_i g^\dagger + g\,\partial_i \delta g^\dagger$$

$$= \delta g(\partial_i g^\dagger) + g\,\partial_i\left(-g^\dagger \delta g\, g^\dagger\right)$$

$$= \delta g(\partial_i g^\dagger) - g\,\partial_i g^\dagger \delta g\, g^\dagger - g g^\dagger \partial_i \delta g\, g^\dagger - g g^\dagger \delta g\, \partial_i g^\dagger$$

$$\left\{\begin{array}{l} \delta(gg^\dagger) = 0 \\ g\,\delta g^\dagger = -\delta g\, g^\dagger \\ \delta g^\dagger = -g^\dagger \delta g\, g^\dagger \end{array}\right. \quad \cdots \text{(A)}$$

$$= \cancel{\delta g(\partial_i g^\dagger)} - g\,\partial_i g^\dagger \delta g\, g^\dagger - \cancel{g g^\dagger}\, \partial_i \delta g\, g^\dagger - \cancel{\delta g\, \partial_i g^\dagger}$$

$$= -g\left(\partial_i g^\dagger \delta g + g^\dagger \partial_i \delta g\right) g^\dagger$$

$$= -g\,\partial_i\left(g^\dagger \delta g\right) g^\dagger \quad \cdots \text{(1)}$$

Let's write out $\varepsilon^{ijk}\,\mathrm{Tr}\left[(g\partial_i g^\dagger)(g\partial_j g^\dagger)(g\partial_k g^\dagger)\right] = Z$ explicitly now:

$$\delta Z = \varepsilon^{ijk}\,\mathrm{Tr}\left[(g\partial_i g^\dagger)(g\partial_j g^\dagger)\,\delta(g\partial_k g^\dagger)\right]$$

$$= -\varepsilon^{ijk}\,\mathrm{Tr}\left[(g\partial_i g^\dagger)(g\partial_j g^\dagger)\, g\,\partial_k (g^\dagger \delta g) g^\dagger\right] \text{(using (1))}$$

As written earlier, this is clearly also invariant under coordinate change of form $x \longrightarrow x + \delta x$ because we could think of $g \to g + \delta g$ (which we proved invariant) as inducing some coordinate change, as well.

3) a $\langle \theta' | H | \theta \rangle = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-im\theta'} \langle m | H | n \rangle$

where $\langle m | H | n \rangle \sim e^{-|n-m| S_0}$

$$= \sum_{\substack{m=-\infty \\ m=n}}^{\infty} e^{im\theta} e^{-im\theta'} \langle m | H | m \rangle + \phi$$

$$= \sum_{m} e^{im(\theta-\theta')} e^{0} + \phi$$

$$= \delta(\theta-\theta') + \sum_{m} \sum_{n} e^{in\theta} e^{-im\theta'} \langle m | H | n \rangle$$

Now, in the second term, we can have $m = n \pm 1$, $m = n \pm 2$ and so on.. but we can always change the limits of summation to ensure we pick a δ-function

$$= \delta(\theta-\theta') \left[1 + e^{-s} \cos\theta + e^{-2s} \cos 2\theta + \cdots \right]$$

Note that since $S_0 \gg 1$, terms apart from $m = n$ will be i.e $e^{-|m-n| S_0} \ll 1$ when $m \neq n$, and vanish as difference increases.

Hence,

$$\langle \theta' | H | \theta \rangle = \delta(\theta-\theta') \left[\text{Some Energy Eigenvalue for ex: } 1 + e^{-s} \cos\theta + \cdots \right]$$

3. a. $\langle \theta' | H | \theta \rangle = \sum_{m,n} e^{-im\theta'} e^{+in\theta} \langle m | H | n \rangle$

Let $n \to m + \xi$

$$= \sum e^{im\theta} e^{i\xi\theta} e^{-in\theta'} f(m-n)$$

$$= \sum_m e^{im(\theta-\theta')} \sum_\xi e^{i\xi\theta} f(-\xi)$$

$$= \delta(\theta-\theta') \, f(\theta)$$

where $f(\theta) = \sum_\xi e^{i\xi\theta} f(-\xi)$ is the energy eigenvalue

Hence $\langle \theta' | H | \theta \rangle \propto \delta(\theta - \theta')$ — ①

b) We note that $e^{-H\tau} \approx \mathbb{1} - H\tau + \dots$

where $H\tau \ll 1$

Let's replace H by $\mathbb{1}$ in ①

$$\langle \theta' | \mathbb{I} | \theta \rangle \propto \delta(\theta - \theta')$$

$$\Rightarrow \approx \langle \theta' | e^{-H\tau} | \theta \rangle \propto \delta(\theta - \theta')$$

We can calculate average value of $e^{-H\tau}$ over all θ-vacua.

Comparing with L.H.S, we see that

$$H\tau = -2e^{-S_0}\cos\theta$$

$$\boxed{H \sim -\cos\theta}$$

Note : I could never have figured this negative sign without useful reference found on John Preskill's CalTech website !