Quantum Field Theory - Assignment 5

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Problem 1 Srednicki 16.1

Solution : Attached hand written at the end.

Problem 2 Srednicki 22.1

Solution : The Noether current is defined as :

$$j^{\mu}(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial(\partial_{\mu}\varphi_a(x))}\delta\varphi_a(x)$$

And then the Noether charge, Q is defined as :

$$Q \equiv \int d^3x \;\; j^0(x)$$

$$Q \equiv \int d^3x \;\; \frac{\partial \mathcal{L}(x)}{\partial(\partial_0\varphi_a(x))}\delta\varphi_a(x)$$

$$Q \equiv \int d^3x \;\; \frac{\partial \mathcal{L}(x)}{\partial\dot{\varphi}_a(x)}\delta\varphi_a(x)$$

$$Q = \int d^3x \;\; \Pi^a(x)\delta\varphi_a(x) \tag{1}$$

We have to evaluate $[\varphi_a, Q]$

$$[\varphi_a, Q] = [\varphi_a, \int d^3x \;\; \Pi^a(x)\delta\varphi_a(x)] \tag{2}$$

$$= \int d^3x \;\;\; [\varphi_a, \Pi^b(x)]\delta\varphi_b(x)\delta_{ab} \tag{3}$$

Using the commutation relation,

$$[\Pi(\mathbf{x}, t), \varphi(\mathbf{x}', t)] = -i\delta^3(\mathbf{x} - \mathbf{x}')$$

Evaluating (3) and doing the δ integration, we get

$$[\varphi_a, Q] = i\delta\varphi_a$$

Problem 3 : Consider the Lagrangian for N complex scalar fields ϕ^a, with a $= 1, 2, ...N$:

$$\mathcal{L} = \partial_\mu \phi_a^\dagger \partial^\mu \phi^a - V(|\phi|)$$

with $|\phi^2| = \sum_a \phi_a^\dagger \phi^a$

a) Find the global symmetries of $\mathcal{L}$?
b) Use Noether's theorem to compute the conserved currents associated with this symmetry ?
c) Construct the charges. In quantum theory, find their commutators ?

Solution : This given Lagrangian is invariant under global U(N) symmetry. It is because the lagrangian consists of N scalar fields and their adjoint and such a unitary group transformation leaves the lagrangian invariant. The symmetry is global since the parameter α which we define below does not depend on space-time coordinates.
The scalar fields transforms as

$$\phi_a = \mathcal{U}_a{}^b \phi_b$$

where $\mathcal{U}$ is given by :

$$\mathcal{U} = e^{i\alpha^m \mathcal{T}_m}$$

$$\phi'(x) = e^{i\alpha^m \mathcal{T}_m} \phi(x) \tag{4}$$

where, $\mathcal{T}$ are N $\times$ N hermitian matrices (generators of the U(N) group).

Note that any $N\times$ N unitary matrix can be written in terms of hermitian matrix $\mathcal{T}$ as :

$$U = \exp\left(i\alpha^k \mathcal{T}_k\right)$$

To convince ourselves, we can check that :

$$\exp\left(i\alpha^k \mathcal{T}_k\right)^\dagger \exp\left(i\alpha^k \mathcal{T}_k\right) = 1 = UU^\dagger$$

We used the fact that $\mathcal{T}$ was a hermitian matrix.
This symmetry group has N^2 generators. This means that we will have N^2 conserved currents and charges. It is important to note that the symmetry group related to EM interactions is U(1) which has just a single generator. Also, U(N) can be thought as $U(N) = U(1) \bigotimes SU(N)$. The U(1) group has one generator (photon !) and SU(N) has $N^2 - 1$ generators and they add up neatly to N^2.

$$\phi'(x) \equiv \phi(x) + i\alpha\mathcal{T}\phi(x)$$

This gives us,

$$\delta\phi'(x) = i\alpha\mathcal{T}\phi(x)$$

Now, the Noether current is given by :

$$j^\mu = \frac{\partial L}{\partial(\partial_\mu \phi)} \delta\phi'(x)$$

$$j^\mu = i\alpha \frac{\partial L}{\partial(\partial_\mu \phi)} T\phi(x)$$

To press the fact that we have same number of currents as generator, we specify it with an index.

$$j_i^\mu = i\alpha \frac{\partial L}{\partial(\partial_\mu \phi)} T_i\phi(x)$$

The Noether charge is given by :

$$Q_i \equiv \int d^3x \, j_i^0$$

This gives us,

$$Q_i \equiv i\alpha \int d^3x \, \Pi \, T_i \phi(x)$$

Now, we also recall our old commutation relation,

$$[\Pi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = -i\delta^3(\mathbf{x} - \mathbf{x}')$$

Corrected Part(b) and Part(c) on the next page.

Problem 4 Srednicki 36.5

Solution : N massless Weyl fields ψ_j

$$\mathcal{L} = i\psi_j^\dagger \sigma^\mu \partial_\mu \psi_j$$

where the repeated index j is summed. The lagrangian is clearly invariant under the U(N) transformation,

$$\psi_j \to U_{jk}\psi_k$$

where U is a unitary matrix. State the invariance group of the following cases :

a) N Weyl fields with a common mass m,

$$\mathcal{L} = i\psi_j^\dagger \sigma^\mu \partial_\mu \psi_j - \frac{1}{2}m\left(\psi_j\psi_j + \psi_j^\dagger\psi_j^\dagger\right)$$

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Let's make the transformation :

$$\psi_j \to A_{ji}\psi_i$$

where, A will be identified belonging to the group such that the lagrangian is invariant under the above change of ψ.

$$\mathcal{L} \to i(A_{ji}\psi_i)^\dagger \sigma^\mu \partial_\mu (A_{ji}\psi_i) - \frac{1}{2}m(A_{ji}\psi_i)(A_{ji}\psi_i) - \frac{1}{2}m(A_{ji}\psi_i)^\dagger(A_{ji}\psi_i)^\dagger$$

$$\mathcal{L} \to i(\psi_i^\dagger A_{ji}^\dagger \sigma^\mu \partial_\mu A_{ji}\psi_i) - \frac{1}{2}m(\psi_i A_{ij}^T A_{ji}\psi_i) - \frac{1}{2}m(\psi_i^\dagger A_{ij}^* \psi_i^\dagger A_{ji}^\dagger)$$

Now, is clear to see that the invariant transformation must be real matrices since we have a complex conjugate in the second part which cannot be cancelled if we don't consider it to be real.

This changed lagrangian will give the initial if following is true :

- $A = A^*$
- $A^\dagger A = \mathcal{I}$
- $AA^T = \mathcal{I} = A^T A$

These are the properties of real unitary matrix which is all called orthogonal. **Hence this is invariant under O(N)**

b) N massless Majorana fields,

$$\mathcal{L} = \frac{i}{2}\Psi_j^T \mathcal{C}\gamma^\mu \partial_\mu \Psi_j$$

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We can re-write the given Lagrangian in the following form :

$$\mathcal{L} = \frac{i}{2}\bar{\Psi}_j\gamma^\mu\partial_\mu\Psi_j$$

We have used the result that $\Psi^T\mathcal{C} = \bar{\Psi}_j$

Now, using Srednicki (36.27) and neglecting the **boundary term**, we obtain :

$$\mathcal{L} = \frac{i}{2}\left[\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \zeta^\dagger\bar{\sigma}^\mu\partial_\mu\zeta\right]$$

Now, we recall that for Majorana fields, particles are their own anti-particles and we obtain :

$$\mathcal{L} = i\left(\chi_j^\dagger\sigma^\mu\partial_\mu\chi\right)$$

Comparing the above with the lagrangian for N-massless Weyl fields, we see that they are the same. Hence, they transform similarly. **They transform under U(N)**.

c) N Majorana fields with a common mass m,

$$\mathcal{L} = \frac{i}{2}\Psi_j^T\mathcal{C}\gamma^\mu\partial_\mu\Psi_j - \frac{1}{2}m\Psi_j^T\mathcal{C}\Psi_j$$

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This case is similar to the situation in part(a) where, the we can do similar stuff with what we did in the previous part. This lagrangian will be invariant under same group as in part(a). This is **invariant under O(N)**.

d) N massless Dirac fields

$$\mathcal{L} = i\bar{\Psi}_j\gamma^\mu\partial_\mu\Psi_j$$

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Each Dirac field consists of two Weyl spinors. This lagrangian is that of 2N massless Weyl fields and it remains invariant under U(2N). Also, note that in part(b), the lagrangian was invariant under U(N) since they were Majorana massless fields where particle and antiparticle were same. But, here there is an increase of 2-fold symmetry because of the mixing and we have a lagrangian that is invariant under **U(2N)** (enlarged group).

e) N Dirac fields with common mass m,

$$\mathcal{L} = i\bar{\Psi}_j\gamma^\mu\partial_\mu\Psi_j - m\bar{\Psi}_j\Psi_j$$

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This is now almost identical to Part(c) done above for N massless Majorana fields. Here, we will have the **symmetry group as O(2N)** since, there are twice as many terms, infact 2N-massive Weyl spinors.

Problem 5 Srednicki 33.2

The Lorentz group (homogenous) consists of boost and rotation. If **K** and **J** are the boost and the angular momentum operators, then we can write :
The Lie algebra for this $SO(3,1)$ homogenous Lorentz group is :

$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k$$

We define N_i and $N_i^\dagger$ as below :

$$N_i = \frac{1}{2}(J_i - iK_i)$$

$$N_i^\dagger = \frac{1}{2}(J_i + iK_i)$$

Now, the commutation relations of N_i and $N_i^\dagger$ are,

$$[N_i, N_j] = i\epsilon_{ijk}N_k \tag{5}$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger \tag{6}$$

$$[N_i, N_j^\dagger] = 0 \tag{7}$$

This implies that

- **N** and $\mathbf{N}^\dagger$ are independent and have intrinsic SU(2) symmetry.

And Lorentz group has SU(2) $\bigotimes$ SU(2) symmetry (same algebra) following this with 6 independent parameters (i.e 3 for boosts and 3 for angular momentum).

We have to prove Eq. 1-3 using the above defined relations.

Proof 1 :

$$[N_i, N_j] = \frac{1}{4}[J_i - iK_i, J_j - iK_j]$$

$$4[N_i, N_j] = [J_i, J_j] + [J_i, -iK_j] - i[K_i, J_j] - [K_i, K_j]$$

This gives us using above defined relations

$$4[N_i, N_j] = i\epsilon_{ijk}\left(J_k - iK_k - iK_k + J_k\right)$$

$$4[N_i, N_j] = 4i\epsilon_{ijk}N_k$$

$\Longrightarrow$

$$[N_i, N_j] = i\epsilon_{ijk} N_k$$

<u>Proof 2</u> :

$$[N_i^\dagger, N_j^\dagger] = \frac{1}{4}[J_i + iK_i, J_j + iK_j]$$

$$4[N_i^\dagger, N_j^\dagger] = [J_i, J_j] + [J_i, iK_j] + i[K_i, J_j] - [K_i, K_j]$$

This gives us using above defined relations

$$4[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}\left(J_k + iK_k + iK_k + J_k\right)$$

$$4[N_i, N_j] = 4i\epsilon_{ijk} N_k^\dagger$$

$\Longrightarrow$

$$[N_i, N_j] = i\epsilon_{ijk} N_k^\dagger$$

<u>Proof 3</u> :

$$[N_i, N_j^\dagger] = \frac{1}{4}[J_i - iK_i, J_j + iK_j]$$

$$4[N_i, N_j] = [J_i, J_j] + [J_i, iK_j] - i[K_i, J_j] + [K_i, K_j]$$

This gives us using above defined relations

$$4[N_i, N_j] = i\epsilon_{ijk}\left(J_k + iK_k - iK_k - J_k\right)$$

$$4[N_i, N_j] = 0$$

$\Longrightarrow$

$$[N_i, N_j] = 0$$

<u>Problem 6</u> <u>Srednicki 34.2</u>

<u>Solution</u> : We need to show that Eq (34.9) and Eq (34.10) obey Eq (34.4)

$$(S_L^{ij})_a{}^b = \frac{1}{2}\epsilon_{ijk}\sigma_k \quad (34.9)$$

$$(S_L^{k0})_a{}^b = \frac{1}{2}i\sigma_k \quad (34.10)$$

$$[S_L^{\mu\nu}, S_L^{\rho\sigma}] = i\left(g^{\mu\rho}S_L^{\nu\sigma} - g^{\nu\rho}S_L^{\mu\sigma} - g^{\mu\sigma}S_L^{\nu\rho} + g^{\nu\sigma}S_L^{\mu\rho}\right) \quad (34.4)$$

Writing in terms of same indices as we need to prove, 34.4 reads as :

$$[S_L^{ij}, S_L^{mn}] = i\left(g^{im}S_L^{jn} - g^{jm}S_L^{in} - g^{in}S_L^{jm} + g^{jn}S_L^{im}\right)$$

Now, using (34.9) and taking the commutator with i, j & m, n indices, we have :

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\epsilon^{ijk}\epsilon^{mnp}[\sigma_k, \sigma_p]$$

Now if we have $k = p$ above then, the commutator will vanish. But, we can play around with taking $k = m$ and $k = n$. This gives us,

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(\epsilon^{ijm}\epsilon^{mnp}[\sigma_m, \sigma_p] + \epsilon^{ijn}\epsilon^{mnp}[\sigma_n, \sigma_p]\right)$$

Now doing the same thing with using $p = i$ and $p = j$ we obtain eventually,

$$[S_L^{ij}, S_L^{mn}] = \frac{1}{4}\left(\epsilon^{ijm}\epsilon^{mni}[\sigma_m, \sigma_i] + \epsilon^{ijm}\epsilon^{mnj}[\sigma_m, \sigma_j] + \epsilon^{ijn}\epsilon^{mni}[\sigma_n, \sigma_j] + \epsilon^{ijn}\epsilon^{mnj}[\sigma_n, \sigma_j]\right) \tag{8}$$

Now, this looks pretty much set for matching what we expect.
Recall that,

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma^c = 4iS_L^{ab}$$

Using this in Eq(8) and noting that the delta-function that comes in can be written as metric tensor. We obtain, the following :

$$[S_L^{ij}, S_L^{mn}] = i\left(g^{im}S_L^{jn} - g^{jm}S_L^{in} - g^{in}S_L^{jm} + g^{jn}S_L^{im}\right)$$

This concludes Part(a).

<u>Part B</u>: Show that

$$(S_L^{k0})_a{}^b = \frac{1}{2}i\sigma_k \quad (34.10)$$

obeys 34.4 as well.
We take the commutator constructed of (34.10) as,

$$[S^{k0}, S^{l0}] = -\frac{1}{4}[\sigma_k, \sigma_l]$$

Using the commutator of $\sigma's$ we get,

$$[S^{k0}, S^{l0}] = \frac{-i}{2}\epsilon_{klm}\sigma_m = -iS_L^{kl} \tag{9}$$

Now, we go back to (34.4) and see what we should expect from that in this case.

$$[S^{k0}, S^{l0}] = i\left(g^{kl}S_l^{00} - g^{0l}S_L^{k0} - g^{k0}S_L^{0l} + g^{00}S_L^{kl}\right)$$

Note the following things : 1) S is antisymmetric tensor and 2) Metric is diagonal. The first term is zero, second and third are zero too ! And, these reduce the above to :

$$[S^{k0}, S^{l0}] = -iS_L^{kl} \tag{10}$$

Comparing (9) and (10), we prove the required.