Realm of Topological Field Theory

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Abstract

The extensive study of topological field theories has not only resulted in our improved understanding of gauge theories and supersymmetry but also in elegant proofs of many mathematical results. The stochastic quantization of Langevin equation, BRST symmetry, twisted supersymmetry and Nicolai map are all intertwined with the idea of topology. Since these theories have no dynamical degrees of freedom and all the excitations are purely topological, it is hoped that this will prove crucial in our understanding of the confinement phase of QCD and many other phenomenon. In this report, we will discuss some properties of the topological theories and elaborate the relation between supersymmetry and topological field theories. We will also briefly talk about cases of open gauge algebras and reducible theories and point out the limitations of the usual BRST quantization approach and the following requirement for a more generalized scheme of quantization based on introduction of anti-fields developed by Batalin and Vilkovisky.

Contents

1 Introduction

Topology, in the mathematics literature, is defined through the notion of homeomorphism equivalence class. Two manifolds $\mathcal{M}$ and $\mathcal{M}^{'}$ are said to be homeomorphic if there exists a homeomorphism f : $\mathcal{M} \mapsto \mathcal{M}^{'}$ [1]. One can then divide the manifold into homeomorphism equivalence classes. An object which takes on a constant value on each class is called "topological invariant".However, we can also further subdivide each homeomorphism class into diffeomorphism (i.e C^{∞} mappings) between the members. An object which is invariant under metric deformations (i.e topological) is certainly also diffeomorphism invariant[1].

1.1 Definitions

In BRST quantization of gauge theories, one constructs a BRST operator Q which is nilpotent. The variation of any functional $\mathcal{O}$ is denoted by $\delta\mathcal{O} = \{Q, \mathcal{O}\}$, where the bracket is used to represent the graded commutator with the fermionic charge Q. A state which is annihilated by Q is called Q-closed, while a state of the form $Q|\chi\rangle$ is called Q-exact. From the BRST invariance of the vacuum, we can conclude that the vacuum expectation value of Q$\mathcal{O}$ for any functional $\mathcal{O}$ is zero i.e

$$\langle 0|\mathcal{O}|0\rangle = 0$$

An operator of the form {Q,$\mathcal{O}$} is called BRST commutator. The energy-momentum tensor $T_{\alpha\beta}$ is defined as the change in action under smooth deformations of the metric.

$$\delta_g S = \frac{1}{2}\int_M d^n x\sqrt{g}\delta g_{\alpha\beta} T_{\alpha\beta} \tag{1}$$

We assume throughout that the functional measure in the path integral is both Q-invariant and metric independent. If it is not the case, we have to check for metric anomalies which is outside the scope of this report.

A topological field theory (TFT) is characterized by following :

- Collection of fields defined on a Riemannian manifold($\mathcal{M}$,g)
- A nilpotent operator which is Grassmann odd
- Physical states are in Q-cohomology class.
- Energy-momentum tensor is Q-exact i.e

$$T_{\mu\nu} = QG_{\mu\nu}$$

[1] This means that f and f^{-1} are continuous mappings

Q is referred to as 'BRST charge' (also BRST operator) and the Grassmann grading corresponds to the ghost number.

In general, Q is metric independent and is the simplest situation. However, there are far more interesting cases where $T_{\alpha\beta}$ is BRST commutator even when Q fails to be metric independent (SUSY QM and Sigma models). There also exists cases where $T_{\alpha\beta}$ is not even a BRST commutator, though, it is possible even then, sometimes, to establish the topological nature.
Let's consider the change in the partition function :

$$Z = \int \mathcal{D}\phi e^{-S_q} \tag{2}$$

under an infinitesimal change in the metric, we get :

$$\begin{aligned}
\delta_g Z &= \int \mathcal{D}\phi e^{-S_q} \left(-\frac{1}{2} \int_M d^n x \sqrt{g} \delta g^{\alpha\beta} T_{\alpha\beta} \right) \\
&= \int \mathcal{D}\phi e^{-S_q} \left(-\frac{1}{2} \int_M d^n x \sqrt{g} \delta g^{\alpha\beta} Q G_{\alpha\beta} \right) \\
&= \int \mathcal{D}\phi e^{-S_q} Q\chi_{\alpha\beta} = \langle Q\chi_{\alpha\beta} \rangle = 0
\end{aligned}$$

where, $\chi = -\frac{1}{2}\int_M d^n x \sqrt{g} \delta g^{\alpha\beta} G_{\alpha\beta}$. We have just used the fact that vacuum is BRST invariant. This means that the partition function does not depend on the local structure of the manifold : Z is a topological invariant.

$$S(\phi) = \int d^d x \sqrt{g} \left[g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] \tag{3}$$

where $g^{\mu\nu}$ is the Riemannian metric and ∇ is the covariant derivative.

We can define the energy-momentum tensor as :

$$T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}$$

We can now write it as [2] :

$$\langle T_{\mu\nu} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \left(\frac{\delta S}{\delta g^{\mu\nu}} \right) \exp\left[\frac{-S(\phi)}{\hbar} \right]$$

$$\langle T_{\mu\nu} \rangle = 0$$

Having established that the metric variation of the partition function vanishes,and in turn that the energy-momentum tensor is zero, we now look whether we have other metric independent correlation functions in the theory ?

[2]Using the fact that Z is already independent of metric

We need to examine the presence of other metric independent correlation functions in the theory. Let's start by considering the vacuum expectation value of an observable :

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi e^{-S} \mathcal{O}(\Phi)$$

We have to derive the conditions for this execration value to be zero.

$$\delta \langle \mathcal{O} \rangle = \int \mathcal{D}\Phi e^{-S} (\delta_g \mathcal{O} - \delta_g S_q \cdot \mathcal{O}) \tag{4}$$

Assume that $\mathcal{O}$ satisfies the following properties :

$$\delta_g \mathcal{O} = QT \ \ , \quad Q\mathcal{O} = 0$$

for some T, we then have :

It is interesting to note that even though χ which depends on $V_{\alpha\beta}$ contains metric dependence, we have wrapped it up in form of BRST commutator and still have metric independence.

TFT can be classified in two types : 1. Schwarz type 2. Cohomological type (Witten-type).

1.2 Schwarz Type

The classical action is explicitly independent of the metric. Chern-Simons theory is a prototype of this class of topological field theories introdced by Witten in 1980's. The metric independence implies that the classical stress-energy tensor of TFT vanishes.In addition, even the quantum stress-energy tensor vanishes because of the fact that the remaining part of quantum action has been recasted as a BRST commutator. $\frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu} = 0$. The alternate cases where the classical action depends explicitly on metric is not dealt here. It is also clear from the equation below that the quantum action for Schwarz type theories do not enjoy the property that the quantum action is Q-exact.

$$S_q(\phi, g) = S_c(\phi) + QV(\phi, g)$$

1.3 Witten Type

In Witten-type topological field theories, the topological invariance is more subtle. The lagrangian generally depends on the metric explicitly, but one shows that the expectation value of the partition function and special classes of correlation functions are diffeomorphism [3] invariant.

The important characteristic of Witten-type theory is that the quantum action S_q, which comprised the classical action plus all necessary gauge fixing and ghost terms, can be written as BRST commutator i.e.,

[3]Roughly speaking, this means that they are metric independent

$$S_q = QV$$

for some functional $V(\phi, g)$ of the fields and Q is nilpotent.

If we consider a conventional gauge field theory, for example Yang-Mills theory and especially the BRST symmetry, we know that corresponding to a local symmetry one can construct a BRST operator Q, which is nilpotent i.e $Q^2 = 0$. The field content is given by Φ and the variation of any functional of the fields Φ is denoted by $\delta\mathcal{O} = Q, \mathcal{O}$.

It is well established that global supersymmetric theories possess a topological invariant $\Delta = n_B - n_F$, which measures the difference between the number of bosonic and fermonic zero energy states. We will see that there exists a intimate connection between the index and the Nicolai map. The index turns out to the the winding number [4] of the map.

In theories with a global supersymmetry there exists a mapping (generally, non-local) of the bosonic fields whose determinant cancels the Pfaffian (Salam-Mathews determinant) of the fermionic fields present. This existence of the 'Nicolai Map' is central to the idea of implementing models in a SUSY preserving way on lattice. In fact, as shown in [**?**] it is also possible to formulate supersymmetry on a discrete space time lattice by preserving Nicolai map as substitute to SUSY algebra. Let us consider the SUSY QM Lagrangian :

$$L = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}P'(\phi)^2 + \psi_i\frac{d\psi_i}{dt} + i\psi_1\psi_2 P''(\phi)$$

where $P(\phi)$ is a super potential.

If we consider the mapping (called 'Nicolai mapping') from ϕ to ξ as ,

$$\xi = \frac{d\phi}{dt} + P'(\phi)$$

,

We observe that the Jacobian of this change of variables i.e $\frac{\delta\xi}{\delta\phi}$ exactly cancels the fermionic determinant and thus the effective lagrangian for ξ becomes gaussian except a total derivative term that can be neglected owing to periodic boundary conditions. The partition function then takes the form,

$$Z = \int \mathcal{D}\xi \;\, e^{-\sum_x \xi^2/2} \tag{5}$$

This has an immediate advantage. T he form of the super potential has disappeared from 'Z' and hence it cannot depend on any coupling constants in the model. It is *topologically invariant*. It is therefore clear, that the existence of local Nicolai map is intimately related to the presence of a topological symmetry.

[4]Number of times the map covers the space of functions

2 Supersymmetry

The Coleman-Mandula theorem clearly meant that there was no non-trivial way of mixing particles with integer and half-integer spin. Wess and Zumino discovered field theoretic models with this extended symmetry (called 'supersymmetry') which connects Bose and Fermi fields and are generated by charge transforming like spinors under Lorentz group (supercharges). These supercharges give rise to a new system of commutation and anti-commutation relations, which is not precisely a Lie algebra but a graded algebra. This has a $\mathcal{Z}_2$ grading. In 1975, Haag, Lopuszanski & Sohnius showed that the energy-momentum operators appear among the elements of this pseudo Lie algebra which hints that the fusion between internal and space-time symmetries must exist.

As it turns out, supersymmetry is the only non-trivial extension of Poincaré symmetry that is compatible with the general principles of relativistic quantum field theory.

2.1 Extending the algebra

Supersymmetry enlarges the Poincaré algebra by including spinor supercharges :

$$I = 1, ...\mathcal{N} \qquad \begin{cases} Q^I_\alpha \quad \alpha = 1,2 & ; \quad \text{Left Spinor} \\ \bar{Q}_{\dot{\alpha} I} = (Q^I_\alpha)^\dagger & ; \quad \text{Right Spinor} \end{cases} \tag{6}$$

Here, α is a spinor label, and $\mathcal{N}$ is the number of independent supersymmetries of the algebra.

The Poincaré generators P^μ and $M^{\mu\nu}$ are bosonic generators. In supersymmetry, we have added sponsorial generators Q^L_α, $\bar{Q^M_\beta}$, where L, M = 1,2,...$\mathcal{N}$. The $\mathcal{N} = 1$ case is simple supersymmetry and $\mathcal{N} > 1$ is extended supersymmetry.

The complex spinorial generators follow the following algebra :

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} Z^{LM}$$

$$[P, Q] = 0$$

$$[Q^L_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^L_\beta$$

$$\{Q^L_\alpha, \bar{Q^M_\beta}\} = \delta^{LM} \sigma^\mu_{\alpha\beta} P_\mu \tag{7}$$

The last one is the most interesting of these four. It roughly means that the supersymmetric generators are square root of the four-momentum. It also means that combining two supersymmetric transformations (one of each helicity) corresponds to space-time translation. Also, in our discussion we neglect any central charges denoted by Z in the first equation. That then reduces to,

$$\{Q^L_\alpha, Q^M_\beta\} = 0$$

Note in passing that Z's are anti-symmetric and commute with all the generators of the supersymmetric algebra. Hence, they are also called as central charges. In case of $\mathcal{N} = 1$ SUSY, this vanishes. R-symmetry is a global symmetry that transforms the supercharges in a supersymmetric theory. Gauging the global symmetry of a physical system involves introducing new degrees of freedom, referred to as gauge fields, to make the system invariant under localized actions of the symmetry transformation.

2.2 Twisted Supersymmetry

In the 80's, Witten noticed that supersymmetry has a deep relation to topology. The simplest example of such a relation is supersymmetric quantum mechanics, which provides a physical reformulation of Morse theory. Their relation is not obvious because as the degrees of freedom in a topological field theory of Witten type and supersymmetric theory is very different. Witten-type theories have no physical degrees of freedom but the supersymmetric theories have them. Their relation becomes more apparent when we follow a procedure referred to as 'twisting'. This *twisting* procedure can be viewed as a modification of Lorentz transformation properties. The main purpose is to define a supersymmetric theory on a general Riemannian manifold. This process leads to atleast one scalar supercharge unlike the spinor we have before the twisting. The *twisting* can be done through different methods based on how we embed the spin connection in the R-symmetry group of the extended supersymmetric theory.

The internal symmetry group of the $\mathcal{N} = 2$ SUSY is given by : $U(1)_I \times SU(2)_I$. This is the R-symmetry in supersymmetric theories. And the global symmetry group of $\mathcal{N} = 2$ SUSY in 4-d is : SU(2) $\times$ SU(2) $\times$ $U(1)_I \times SU(2)_I$, where the first two are Euclidean rotation group and last two are isospin rotation group.

The twist consists as considering the rotation group as : $SU(2)^{'}_L \times SU(2)_R \times$, where SU(2)' is the diagonal subgroup of $SU(2)_L \times SU(2)_I$

Roughly speaking, in the twisting procedure one first selects one of the two components of the rotation group and then replaces it by the diagonal sum of that component with a SU(2) subgroup of the internal group [5]. In the case of N = 2 this can be done in only one way while for N = 4 there are three possibilities due to Marcus, Vafa & Witten.

3 Covariant Quantization : Special & General Case

Most of the theoretical models describing fundamental interactions have gauge freedom. In the path integral approach, it is necessary to "gauge fix" them to ensure that we

[5] So now, for every rotation in euclidean space, we do a similar rotation in isospin space

don't integrate over unphysical or spurious degrees of freedom. In most of the models, this (gauge invariance) is implemented by BRST procedure. But, there is another, and infact more general formalism which can do this fixing while preserving space-time covariance. It is called BRST-antifield or Batalin-Vilkovisky (BV) formalism. When it comes to the quantization of gauge theories in a lagrangian formalism, the framework of Batalin - Vilkovisky (BV)[6][7][8] appears to be superior to all other available schemes. Not only is this easy to implement but it gives for free, a new canonical structure contained in what is known as the antibracket. With the help of this, the definition of a gauge-fixed quantum action can be formulated by means of one equation known as the 'master equation'.In attempts to unify all fundamental particles and interactions in nature, supergravity models have turned out to be serious candidates for a consistent quantum theory. In most of these supergravity theories, the corresponding graded algebra is closed only modulo the classical equations of motion (i.e closed only on-shell). As a consequence BRST invariance of the effective action constructed is lost. This is not the case for standard Yang-Mills theories since they have a closed gauge algebra. The antifield-BRST formalism is capable of handling all the gauge structures, while the original method was devised only for off-shell closed, irreducible gauge algebras. This wide range of application of the antifield formalism is one of its main virtues. Note that we will refer to the antifield formalism, antifield-BRST and BV formalism to denote the same procedure at different places[9].

3.1 Algebras & Irreducibility

The need for an alternate approach is related to the fact of how the gauge algebras of theories behave. We say that the gauge algebra closes off shell (i.e closed gauge algebra) if the commutator of two gauge transformations ($\delta_1\Phi^i = \mathcal{R}^i_\alpha\zeta^\alpha_1$ and $\delta_2\Phi^i = \mathcal{R}^i_\alpha\xi^\alpha_2$) is again a linear combination of originally introduced gauge transformations denoted by :

$$[\delta_1, \delta_2]\Phi^i = f^\gamma_{\alpha\beta}\mathcal{R}^i_\gamma\xi^\alpha_2\xi^\beta_1$$

The case where they are dependent constitutes the class of open gauge algebras which only closes on-shell. Now, let's talk about the reducibility of these gauge theories.

We start with a classical action $S_c(\Phi^i)$ which depends on some fields, denoted by Φ^i with the local symmetry transformations :

$$\delta\Phi^i = R^i_\alpha(\Phi)\epsilon^\alpha \tag{8}$$

Here, ϵ^α denotes the local infinitesimal parameters [6] . If $\delta\Phi^i = 0$ for some non-zero ϵ^α, then the transformations in Eq.(8) are referred to as 'first stage reducible' or just reducible. One can also describe this as a situation where the gauge algebra has zero modes. It is then apparent that if one gauge fixed the theory according to Faddeev-Popov, the determinant will have zero modes. This, then, behaves like a residual

[6]Using De-Witt's condensed notation. See Appendix A

gauge symmetry in the ghost action. We need to further fix this term by introducing 'ghost-for-ghost' phenomenon. It is in this case that to incorporate all the terms it is possible to resort to the Batalin-Vilkovisky machinery, which is guaranteed to produce a BRST invariant quantum action, together with an on-shell nilpotent BRST charge Q. Moreover, it may also turn out that the residual gauge symmetry of the ghost action itself has a zero mode ; if this is the case, the theory is said to be 'second stage reducible' and so on. Alternatively, if $\delta\Phi^i = 0$ implies that $\epsilon^\alpha = 0$, then the transformations in Eq.(8) are referred to as 'irreducible', which means that they are on-shell independent.

The Batalin-Vilkovisky (BV) procedure constructs an action which is suitable for quantization while maintaining proper gauge invariances. BRST symmetry can be retrieved from this and most of the properties follow from the fact that this operator is nilpotent. The price we pay for achieving nilpotent nature of BRST operator is by doubling the initial degrees of freedom by introducing antifields.

$$(S,S) = \frac{\partial_r S}{\partial \Phi^A}\frac{\partial_l S}{\partial \Phi^*_A} - \frac{\partial_r S}{\partial \Phi^*_A}\frac{\partial_l S}{\partial \Phi_A} \tag{9}$$

Here, ∂_r and ∂_l denote the right and left derivatives, respectively. We will explicitly explain the meaning of these derivatives in the next part.

3.2 Batalin-Vilkovisky formalism

Let Z be any superspace. The superspace $Z \bigoplus Z^*$ is naturally endowed with an odd non-degenerate (degree -1) pairing. Let Δ be the Laplace operator associated to this pairing. If χ^i are coordinates on Z (the *fields*) and χ^*_i are the corresponding coordinates on Z^* (the *antifields*), then we have :

$$\Delta = \frac{\partial}{\partial \chi_i^+}\frac{\partial}{\partial \chi^i}$$

The operator Δ is called the Batalin-Vilkovisky Laplacian (it is odd and of order 2) ; note that, if Φ is a homogeneous function in then $\Delta\Phi$ is also homogeneous and $\overline{\Delta\Phi} = \overline{\Phi} + 1 \mod 2$.

The classical master equation for even degree (degree +1) element S (called action) of a Batalin-Vilkovisky algebra is the equation :

$$\Delta^2 = 0$$

Indeed,

$$\begin{aligned}\Delta^2 &= \frac{\partial}{\partial\chi_i^*}\frac{\partial}{\partial\chi^i}\frac{\partial}{\partial\chi_j^*}\frac{\partial}{\partial\chi^j} \\ &= (-1)^{\overline{\chi^i}\cdot\overline{\chi_j^*}+\overline{\chi^i}\cdot\overline{\chi^j}*\overline{\chi_i^*}\cdot\overline{\chi_j^*}+\overline{\chi_i^*}\cdot\overline{\chi^j}}\frac{\partial}{\partial\chi_j^*}\frac{\partial}{\partial\chi^j}\frac{\partial}{\partial\chi_i^*}\frac{\partial}{\partial\chi^i} \\ &= (-1)^{(\overline{\chi^i}+\overline{\chi_i^*})(\overline{\chi^j}+\overline{\chi_j^*})}\frac{\partial}{\partial\chi_j^*}\frac{\partial}{\partial\chi^j}\frac{\partial}{\partial\chi_i^*}\frac{\partial}{\partial\chi^i}.\end{aligned}$$

Since the variables χ^i and χ_i^* have opposite parity, $\overline{\chi^i}+\overline{\chi_i^*} \mod 2 = 1$, for any i. Therefore,

$$\Delta^2 = -\Delta^2$$

i.e., $\Delta^2 = 0$. The cohomology of superspace with respect to the BV-Laplacian is called *BV-cohomology* or Δ-*cohomology*.

Let us consider Φ and Ψ be two homogenous function on the superspace. Then,

$$\begin{aligned}\Delta(\Phi\cdot\Psi) &= \frac{\partial}{\partial\chi_i^+}\frac{\partial}{\partial\chi^i}(\Phi\cdot\Psi) \\ &= \frac{\partial}{\partial\chi_i^+}\left(\frac{\partial\Phi}{\partial\chi^i}\cdot\Psi + (-1)^{\overline{\chi^i}\cdot\overline{\Phi}}\Phi\frac{\partial\Psi}{\partial\chi^i}\right) \\ &= \frac{\partial}{\partial\chi_i^+}\frac{\partial\Phi}{\partial\chi^i}\cdot\Psi + (-1)^{(\overline{\chi^i}+\overline{\Phi})\overline{\chi_i^+}}\frac{\partial\Phi}{\partial\chi^i}\frac{\partial\Psi}{\partial\chi_i^+} + (-1)^{\overline{\chi^i}\cdot\overline{\Phi}}\frac{\partial\Phi}{\partial\chi_i^+}\frac{\partial\Psi}{\partial\chi^i} + \\ &\qquad + (-1)^{(\overline{\chi^i}+\overline{\chi_i^+})\overline{\Phi}}\Phi\frac{\partial}{\partial\chi_i^+}\frac{\partial\Psi}{\partial\chi^i} \\ &= (\Delta\Phi)\cdot\Psi + (-1)^{\overline{\Phi}}\{\Phi,\Psi\} + (-1)^{\overline{\Phi}}\Phi\cdot\Delta\Psi\end{aligned}$$

where $\{\Phi,\Psi\}$ is the so-called BV-bracket, defined by

$$\{\Phi,\Psi\} = (-1)^{\overline{\chi_i^+}\cdot\overline{\Phi}}\frac{\partial\Phi}{\partial\chi_i^+}\frac{\partial\Psi}{\partial\chi^i} - (-1)^{(\overline{\Phi}+1)(\overline{\Psi}+1)+\overline{\chi_i^+}\cdot\overline{\Psi}}\frac{\partial\Psi}{\partial\chi_i^+}\frac{\partial\Phi}{\partial\chi^i}$$

The BV-bracket is best expressed using *left* and *right derivatives*: for a homogeneous vector v in $Z\bigoplus Z^*$, set

$$\begin{aligned}\overrightarrow{\partial}_v\Phi &= \partial_v\Phi \\ \overleftarrow{\partial}_v\Phi &= (-1)^{\overline{v}\cdot\overline{\Phi}}\partial_v\Phi\end{aligned}$$

$\overrightarrow{\partial}$ denotes the right derivative and $\overleftarrow{\partial}$ denotes the left in Eq.(9) and with these notations, the BV-bracket reads

$$\{\Phi,\Psi\} = \frac{\overleftarrow{\partial}\Phi}{\partial\chi_i^+}\frac{\overrightarrow{\partial}\Psi}{\partial\chi^i} - (-1)^{(\overline{\Phi}+1)(\overline{\Psi}+1)}\frac{\overleftarrow{\partial}\Psi}{\partial\chi_i^+}\frac{\overrightarrow{\partial}\Phi}{\partial\chi^i} \tag{10}$$

It can be easily checked that when Φ and Ψ both have bosonic character then Eq.9 is implied directly.

3.3 Derivation of Quantum Master Equation

At the quantum level, the action S can be replaced by a quantum action $W = S + \sum_i \hbar^i M_i$, where the M's are a contribution due to the measure of the path integral. For the transition amplitude to be independent of field variables, we must have that $\exp \frac{i}{\hbar} W$ should be Δ -closed.

$$\Delta \exp \frac{i}{\hbar} W = 0 \tag{11}$$

We can easily conclude using ;

$$\Delta(\alpha\beta) = \alpha\Delta\beta + (-)^{\epsilon_\beta}(\Delta\alpha)\beta + (-)^{\epsilon_\beta}(\alpha, \beta) \tag{12}$$

that the following holds,

$$i\hbar\Delta W = \frac{1}{2}(W, W) \tag{13}$$

Note that when the change in action i.e $\Delta S = 0$, we can taken W = S.
We note that Eq.(13) is the cornerstone of this formalism and is known as "quantum master equation". It ensures that the transition integral is independent of the choice of Ψ.

The solution S of the master equation is the key to this formalism. It can indeed be written explicitly for some cases like Yang-Mills. In these cases, the solution is linear in the anti-fields. In gauge systems with 'open" algebra i.e for which the gauge transformations only close only on-shell, or for on-shell reducible gauge theories, the solution of the master equation is more complicated and can possibly contain terms which are non-lines in anti fields.

It has been known for several years that the anomalies are related to the non-invariance of the functional measure of the path integral under BRST symmetries. In fact, one interesting case where the quantum master equation is violated is indeed in the case of anomalies [2]. Following this, Eq.11 gets modified to :

$$i\hbar\Delta W - \frac{1}{2}(W, W) = \hbar a_\nu c^\nu + \mathcal{O}\hbar^2 \tag{14}$$

Troost et.al showed that the BV formalism provides a suitable setting to discuss anomalies.

$$\mathcal{S} = \int d\tau \left[g_{ij}(u) \left(\frac{du^i}{d\tau}\frac{du^j}{d\tau} + i\bar{\psi}^i \frac{D\psi^j}{D\tau} \right) + \frac{1}{4} R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^i\psi^l - g^{ij}(u)\frac{\partial V}{\partial u^i}\frac{\partial V}{\partial u^j} - \frac{D^2 V}{Du^i Du^j}\bar{\psi}^i\psi^j \right] \tag{15}$$

We consider the superparticle on a Riemannian manifold $\mathcal{M}$ in an arbitrary potential V.

Here, u^i are the coordinates of the Riemannian manifold with metric and curvature denoted by g_{ij} and R_{ijkl} and the ψ^i are the Grassmann odd coordinates of the particle. The covariant derivative is given by :

$$\frac{D}{D\tau} = \frac{x}{j}$$

A De Witt's notation

This notation[10] is used to write the gauge transformations in more compact form :

$$\delta_\epsilon \varphi^i = R^i_\alpha \epsilon^\alpha \mapsto \delta_\epsilon \varphi^i(x) = \int d^n y R^i_\alpha(y,x)\epsilon^\alpha(y) \tag{16}$$

For ex: the transformation of the Yang-Mills gauge field

$$\delta_\epsilon A^a_\mu = D_\mu \epsilon^a = \partial_\mu \epsilon^a + f^a{}_{cb} A^b_\mu \epsilon^c \tag{17}$$

can be written as ,

$$\delta_\epsilon A^a_\mu = R^a_{\mu b}\epsilon^b \quad , \quad R^a_{\mu b}\epsilon^b(x,y) = \partial_\mu \delta(x-y)\delta^a_b + f^a{}_{cb} A^b_\mu \delta(x-y) \tag{18}$$

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