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Supersymmetric Wilson loops on the lattice in the large N limit

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Abstract We propose additional tests of holography by studying supersymmetric Wilson loops in $p+1$-dimensional maximally supersymmetric Yang–Mills (SYM) theories on the lattice in the large N limit. In the dual gravity description, this computation involves calculating the area of a fundamental string worldsheet in certain Type II supergravity backgrounds. Though thermodynamic observables have been computed on the lattice using Monte Carlo methods and agree with the supergravity results in various dimensions, not much has been done for the gauge-invariant operators such as the Wilson loop. We provide analytical predictions for these loops for various non-conformal Dp-brane background cases with $p \leq 2$ in the large N limit and comment on how these can be computed on non-orthogonal lattices in various models.

1 Introduction

The holographic conjecture (the duality between gauge and gravity) has been one of the greatest advances of the 3 decades. It relates a theory of quantum gravity in anti-de Sitter (AdS) spacetime to a supersymmetric gauge theory on the boundary. There were signs of dimensional reduction at play in quantum gravity [1], and a concrete proposal relating Type IIB string theory on AdS$_5$ times S_5 to a four-dimensional superconformal $\mathcal{N} = 4$ SYM was proposed in 1997 and is now known as AdS/CFT conjecture. Soon after, this duality was also extended to admit non-conformal dimensionally reduced versions of $\mathcal{N} = 4$ SYM that are dual to near-extremal, the near-horizon limit of Dp-branes [2] in some well-defined geometry. These ideas were partially motivated by the work of Witten [3] where it was argued that the low-energy theory describing a system of N parallel Dp-branes in flat space is the dimensional reduction of ten-dimensional $\mathcal{N} = 1$ SYM theory down to $p + 1$ dimensions. In the case of N D0-branes, this gives the famous BFSS model [4].

While the four-dimensional gauge theory enjoys several unique properties which facilitate the setting to check holography in detail, such tools cannot be extended to lower dimensions. The lower dimensional gauge theories with maximum supersymmetry content are much more complicated, partly because of their non-integrable and non-conformal nature. Therefore, the availability of a limited set of tools is a major hindrance to exploring holography for a wide class of theories. One of the few known numerical tools which have been used in the last decade to address this issue is to apply the ideas from lattice gauge theory to study the large N, strongly coupled sector of these theories. This application of lattice gauge theory to study supersymmetric Yang–Mills (SYM) theories have undergone substantial progress in the last 2 decades, largely motivated by this goal of understanding holography. It is not the plan to provide an extensive review of this here, but good progress which has resulted is due to a beautiful amalgamation of ideas from twisting of supersymmetric theory to obtain a topological description, idea of integer-form spinors, and some enhanced point group symmetry of non-hypercubic lattices. The case of four dimensions is certainly the most challenging because of its superconformal nature and absence of length scales. Also, the dimensionality and requirement of large N are major problems even with parallelized software which are employed. Hence, it is reasonable to say that the four-dimensional superconformal $\mathcal{N} = 4$ SYM theory on the lattice at large N is still mostly out of reach, but, the lower dimensional theories have been explored and qualitative agreement with the thermodynamics of black branes through holography has been established [5–7]. These numerical computations typically use several million hours of CPU time in order to reliably take the large N limit and perform the continuum extrapolations.

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For the dimensionally reduced theories with dual gravity interpretations, various groups have studied the thermodynamics and thermal phase transitions and have been reasonably successful on the lattice, but much more admittedly remains to be done. One of the drawbacks is that not many observable and correlation functions of gauge-invariant operators have been numerically computed. The eventual goal is to extend these non-perturbative checks to more complicated observables that have not been studied on the lattice. These computations are necessary if one has to claim precision checks of holography across various dimensions. In this work, we consider the Wilson loop operator and argue that we can determine the radius of the black hole dual geometry from this observable in various Dp-brane models. This type of computation was first discussed in [8] using methods based on Gaussian approximation and was later studied using numerical methods for $p = 0$ case in [9] for maximum $N = 17$. It is surprising that this computation has not been performed with improved resources and state-of-the-art methods where $N = 32$ have been reached [10]. The extension of this program to higher dimensional $p + 1$-dimensional SYM theories is not straightforward. This is partly due to the increased computational resources with dimensionality and the lattices used for discretization of these theories, which are not the standard hypercubic lattices.

2 Type II supergravity and Wilson loops

The holographic duality states that at finite temperatures, there is a dual Type IIA (p even) or IIB (p odd) gravity dual description of maximally supersymmetric $p+1$-dimensional SYM in terms of the decoupling limit of Dp-branes. In this article, we will restrict ourselves to $p < 3$. The $p = 3$ case which is conformal has been well studied. For $p > 3$, the supersymmetric gauge theory is non-renormalizable and the computation works in a little different way, see Ref. [2] for those cases. In the large N, strong coupling limit of field theory this reduces to supergravity (SUGRA) where the string frame near horizon, near extremal metric takes the form:

$$ds^2 = \alpha'\Big(\kappa dt^2 + \frac{U^{(7-p)/2}}{\sqrt{d_p\lambda}} dx_p^2 + \kappa^{-1} dU^2 + \frac{\sqrt{d_p\lambda}}{U^{(3-p)/2}} d\Omega_{8-p}^2\Big), \tag{1}$$

where $\kappa = \Big[1 - \Big(\frac{U_0}{U}\Big)^{7-p}\Big] U^{(7-p)/2}/\sqrt{d_p}\lambda$, $d_p = 2^{7-2p}\pi^{\frac{1}{2}(9-3p)}\Gamma\big(\frac{7-p}{2}\big)$ and α' is related to inverse of string tension. The coordinate in the radial direction, U, is usually identified with some energy scale corresponding to the scalars in the SYM theory while the horizon of the dual geometry is at $U = U_0$ and is related through some geometrical factors to black brane temperature.

This dual description breaks down outside some coupling range when $p < 3$ and we can only compute the supergravity predictions for Wilson loop operator when the loop size is big enough [11], however, one can still use the description of classical string worldsheets in some well-defined SUGRA background to compute these loops. The region where it is valid is related to p and given by: $1 \ll \lambda\beta^{3-p} \ll N^{10-2p/(7-p)}$. This can be obtained by demanding that radius of curvature should be large in units of inverse tension (i.e., α') and the string coupling should be small.

The usual Wilson loop is built out of gauge fields, but in supersymmetric gauge theories, a modified gauge-invariant operator was proposed in Refs. [11, 12] which also includes the contribution from the $(9-p)$ adjoint scalars (Φ) in addition to usual gauge fields A. It is defined as follows:

$$W(\mathcal{C}) = \frac{1}{N}\mathrm{Tr}\ \mathbb{P}\ \exp\Bigg[\oint_{\mathcal{C}} d\tau \Big(A_\mu(x)\dot{x}^\mu + \hat{\theta}^i|\dot{x}|\Phi_i(x)\Big)\Bigg], \tag{2}$$

where $\hat{\theta}$ is a unit vector in $\mathbb{R}^{(9-p)}$ in which the probe Dp-brane is separated and $\mathcal{C}$ is the circle (contour) which is parametrized by $x^\mu(\tau)$ and $\mathbb{P}$ is the path ordering. It is normalized such that the large N limit is well-defined. This observable is invariant under supersymmetry variation as required. In fact, there is a well-defined mapping between this Wilson loop operator in gauge theory and the string partition function $Z(\mathcal{C})$ [11, 12] given by

$$\log\langle W(\mathcal{C})\rangle \sim \log Z(\mathcal{C}) \sim -S(\mathcal{C}). \tag{3}$$

On the supergravity side, this computation of the Wilson loop operator is equivalent to either finding the minimal surface which describes the string worldsheet or computing the logarithm of string partition function. The agreement between circular supersymmetric Wilson loops and minimal area in supergravity was one of the first tests of holographic principle [13, 14]. For the circular Wilson loops,[1] we mention the dependence obtained using supergravity calculations for $p < 3$ where generally, $\log(W) \sim t^{-(3-p)/(5-p)}$. Note that for $\mathcal{N} = 4$ SYM this is the well-known $\sqrt{\lambda}$ dependence. We also note that only the $p = 0$ case has yet been verified using numerical simulations [9] while other p remains to be explored on the lattice.

To compute the observable we are interested in, we note that we need to equate the difference in U_0 and U_∞ to the Wilson loop and the mass of W-boson as

[1] If we choose the contour to be a straight line, then the Wilson line operator is a BPS object whose expectation value is precisely equal to unity. However, for a circular loop, it is not equal to one. This is a little counter-intuitive, since a straight line and circle can always be related to a conformal transformation. This shows that the expectation value is not invariant under conformal transformations.

done in Ref. [11]. Once we do this, the computation of log(W), therefore, basically depends on finding U_0, which is computed from the supergravity metric in terms of Hawking temperature T. It is then straightforward to work out the expectation value of the Wilson loop from supergravity and we obtain for $p < 3$:

$$\ln(W(\mathcal{C})) = \frac{1}{2\pi}\left(\frac{(7-p)\ t^{(3-p)/2}}{4\pi\sqrt{d_p}}\right)^{\frac{2}{p-5}}, \qquad (4)$$

where t is the dimensionless temperature constructed using $T = 1/\beta$ and λ in theories with $p < 3$ as $t = T/\lambda^{1/(3-p)}$. Note that this expression reduces to the well-known $\log(W(\mathcal{C})) = 1.89t^{-3/5}$ when $p = 0$. This is our main result which was not yet computed for general p. We also note that strictly the canonical ensemble is not well-defined for $p < 3$ corresponding to the divergence in the IR and related to the evaporation of the Dp-branes as radiation, however this is $1/N$ effect and not seen in the strict planar limit. It is expected that computations done in metastable vacuum configurations would be close to the actual answer and, moreover, this cannot be seen at large N and we can still hope to do a reliable comparison with holography. This is an extra requirement that lattice computations use as large matrices as possible to minimize this effect of IR divergence. In case, it is not possible to numerically study very large N, we have been able to cure these at finite N by adding a mass regulator which stabilizes the numerical calculations and then taking the zero mass limit.

3 Lattice computation

The lattice formulations of various maximally supersymmetric SYM theories in p+1-dimensions (with $p \leq 3$) start similar to the AdS/CFT conjecture with the four-dimensional gauge theory. In order to discretize the $\mathcal{N} = 4$ SYM on the lattice, one performs twisting of field variables to render the theory topological. This results in p-form supercharges and the zero form nilpotent supercharge is the one that is exactly preserved on the lattice. However, this is only a very small fraction of the total supercharges in the target continuum theory. It is hoped that the remaining supersymmetries will follow as we take the continuum limit but this issue is subtle at least in four dimensions and we strongly believe this is not true in practice with a finite number of counter terms. Moreover, the sign problem (related to integration over fermionic variables) which seems under control in lower dimensions at strong couplings with anti-periodic (thermal) boundary conditions return in a rather severe manner in the four-dimensional $\mathcal{N} = 4$ SYM. For these issues and many others, four-dimensional lattice theories have not yet been able to probe holography directly in a well-defined manner. However, we can understand the lattice constructions and holography beyond this four-dimensional case and this is where a lot of progress has been recently made. One of the reasons for this success is that in lower dimensions, the theory is non-conformal and the presence of scale helps the lattice computations. Another reason is that in these lower dimensional cases a greater fraction of the target supersymmetries is exactly kept at finite lattice spacing. These substantially reduce the number of counterterms to be fine-tuned to an extent that it is conjectured that none remains for $p+1$-dimensional SYM theories with $p \leq 2$. Once we do a classical dimensional reduction (i.e., reduce some spatial direction down to a single site on the lattice) of this four-dimensional theory, we end up with $p+1$-dimensional SYM for $p \leq 2$. These theories have been successfully studied close to the planar limit at strong couplings and qualitative agreement with holographic expectations was observed.

While the computation of the Wilson loop was done for the $p = 0$ case, doing this for $p \geq 1$ case is difficult because of the requirement that the lattice discretization retains a nilpotent scalar supercharge and targets the correct continuum limit, it was observed that the non-orthogonal $A_d^\star$ lattices are the natural setting for discretizing those Euclidean d-dimensional maximally supersymmetric YM theories. Though it is possible to consider alternate derivation of Wilson loops in those non-orthogonal geometries, it is complicated. It is complicated because the expressions for Wilson loop has to be computed in non-hyper cubic lattice geometry since the lattice is not an orthogonal (skewed). The edges (or the boundary) can affect the computation similar to what was explored by authors and collaborators in Ref. [5]. Therefore, it is only when it is hypercubic (as discussed in the article) we can compute this in a straightforward manner. It might be possible to compute the Wilson loop for the deformed geometry and it will be dressed by some factor γ which determines the deviation from orthogonal lattice ($\gamma = 0$).

However, it is possible for these supersymmetric theories that one can end up with correct target continuum theory even on an orthogonal geometry (rather than skewed torus) if we can expand the fields around the correct point in the moduli space. For example, it was argued that if all of the $d+1$ links of a $A_d^\star$ lattice is expanded alike i.e., $U = \mathbb{I}$ (up to some powers of lattice spacing) we get the skewed geometry. However, if we rather expand d links around this same vacuum expectation value and the last remaining gauge link around zero, we obtain the target theory on a hypercubic lattice. This corresponds to expanding the action about the asymmetric solution of the moduli equations, see [15] for discussion about this. Though we do not have a rigorous proof to show why such an expansion alters the geometry on which target continuum theory is defined, it is reasonably easy to numerically check that this is true. This can be implemented in the lattice action by adding a term that ensures this special expansion basis. We implemented such a term to carry out some preliminary computations of the critical temperatures and found reasonable agreement for the $p = 1$ case with the conjectured onset of Gregory–Laflamme instability [16]. The location of this transition is determined by the geometry of the lattice and there is a possibility of checking this for a range of γ interpolating between $\gamma = -1/d$ for skewed lattice such as one discussed in Ref. [5] to the orthogonal lattice as discussed in this article ($\gamma = 0$) and we leave this for future work since this is rather involved numerical problem with systematic extrapolations in many parameters and the requirement of large N limit, different aspect ratios, and temperatures (couplings). In addition to the operator discussed here, it will also be interesting to compute the correlation functions of various operators along the lines as described in Ref. [17] to better understand the behavior of non-conformal branes.

4 Summary

We have outlined the procedure to non-perturbatively check the computation of the gauge-invariant non-local supersymmetric Wilson loop (also called Maldacena–Wilson) on the lattice. We have provided expressions for these observables by computing the corresponding observable in the Type II supergravity. It would be desirable to have a rigorous numerical check of this observable on the lattice and hence find agreement with the holographic duality. This would further extend and become a useful addition to the program of verifying gauge/gravity duality using lattice computations of supersymmetric gauge theories across various dimensions which constitutes a first-principle check of gauge/gravity duality.

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