Old *and* new methods *for* new *and* old problems in Physics

COLLOQUIUM AT IIT MADRAS [ONLINE]

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OUTLINE

OLD FOR NEW

- Holography for matrix QM and generic Yang-Mills (YM) theories including AdS/CFT conjecture (circa 1998)

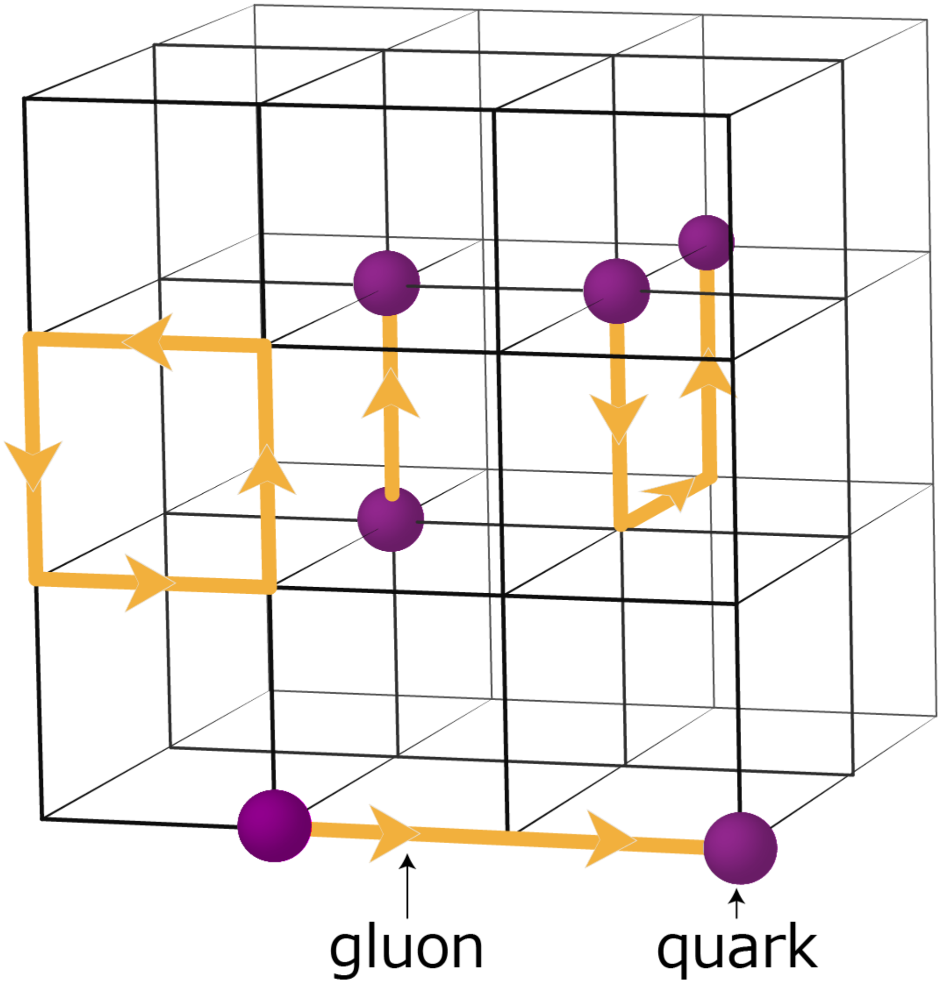
- Lattice studies (Wilson, 1974) of strongly coupled YM theories and applications for holography, dynamical supersymmetry (SUSY) breaking.

NEW FOR OLD

- Tensor network approach to quantum many body systems and its relation to holography and entanglement (circa 2010)

- Applications of this new numerical approach to study Ising model, $O(2)$ models in $2d$ & $3d$ and some lattice gauge theories such as $U(1)$ and $SU(2)$ in lower dimensions.


FERMI—BOSE
BOSE—FERMI


gluon
quark

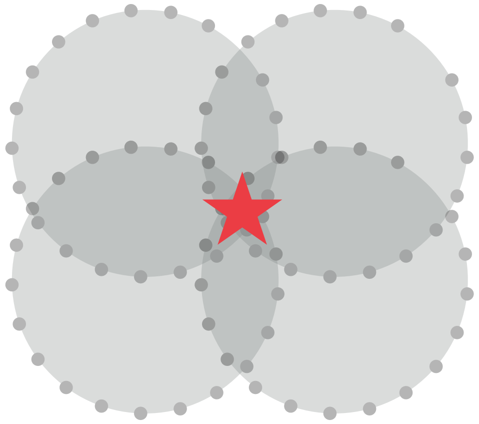
HOLOGRAPHY

The idea that a quantum-gravitational theory in one higher dimension $(d+1)$ is related to some quantum field theory (without gravity) in one lower dimension (d) on its boundary.

It is expected that the theory of quantum gravity will admit a holographic description.

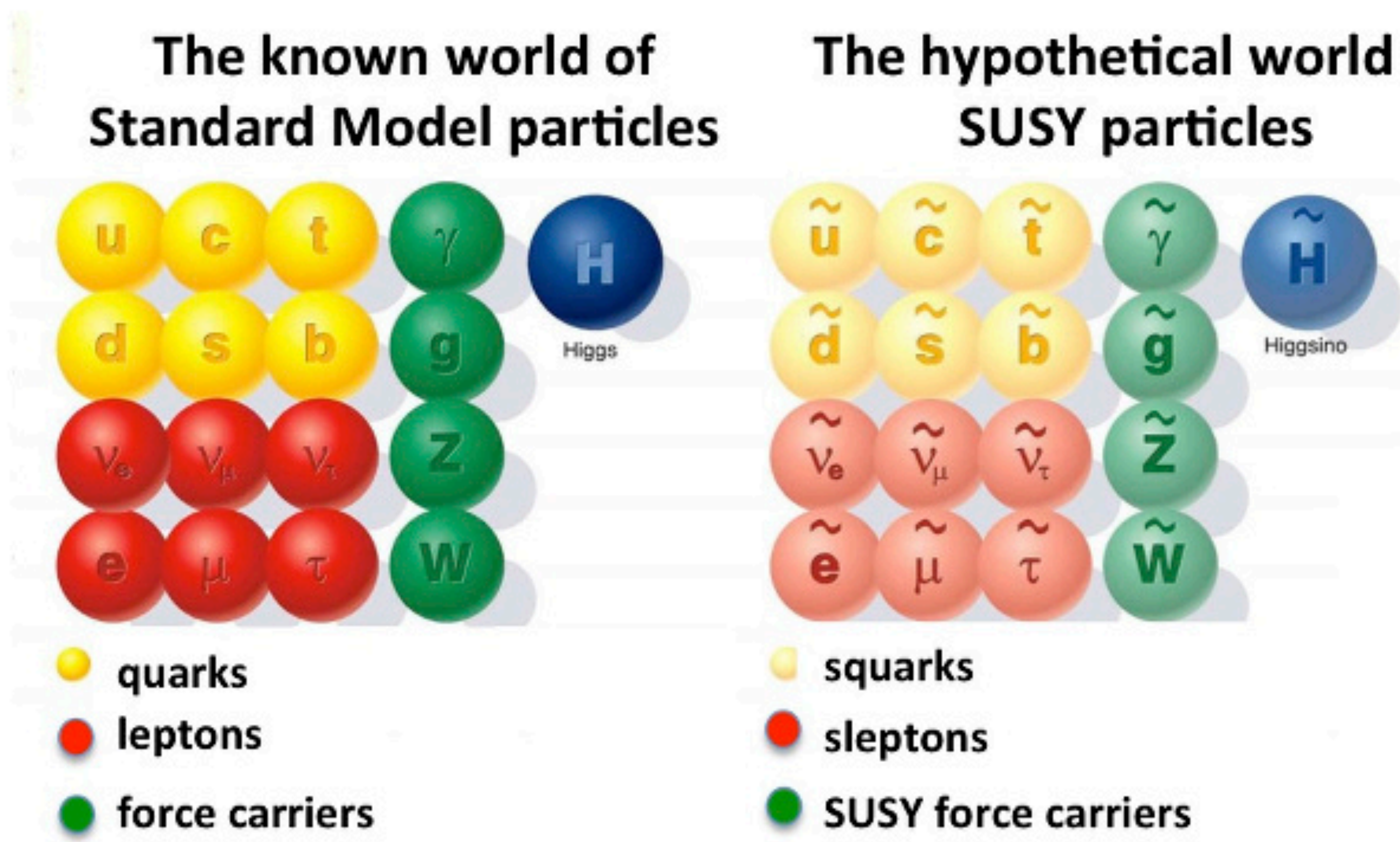
First hints came in 1970s, when Stephen Hawking and Jacob Bekenstein found that the black hole entropy was proportional to the area of its event horizon.

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar}$$



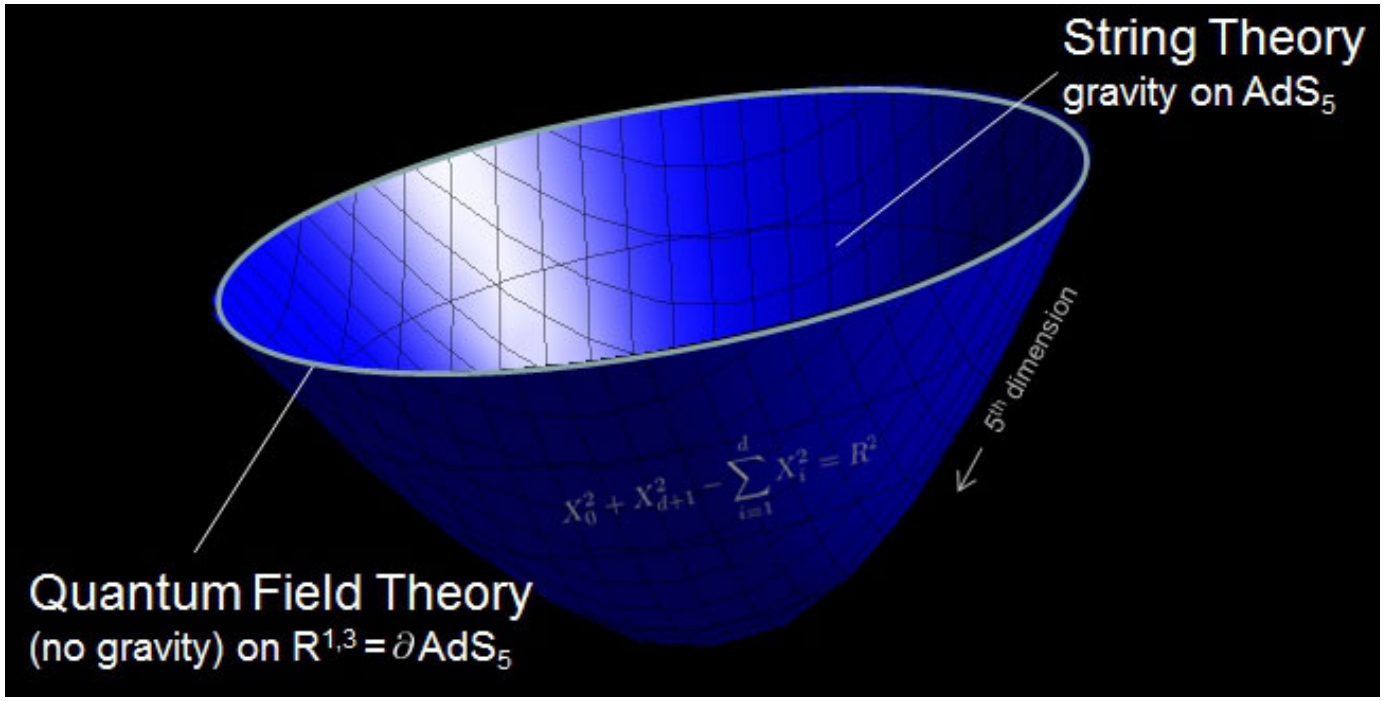
Supersymmetry!

It is elegant, beautiful, broken, and has not been seen until now.



AdS/CFT [Maldacena, 1997]

There is a well-defined correspondence between a five-dimensional quantum theory of gravity in Anti- de Sitter and four dimensional gauge theory on the boundary. In the limit of $N \to \infty, \lambda \gg 1$, the quantum gravity theory is simple Einstein-like gravity in the bulk (low-energy string theory/supergravity [SUGRA]).



Not just AdS_5/CFT_4!

But, there is nothing special about the basic idea of holographic to 4/5 dimensions. Within one year, it was rigorously defined for supersymmetric gauge theories for $d < 4$ even though they are not conformal. One of the most studied cases is AdS_3/CFT_2

Maximally supersymmetric Yang-Mills theory in $p+1$-dimensions also has holographic interpretation at low temperatures in a special limit (large N, strong coupling) in the sense that the SUGRA solutions corresponding to these are black p-brane solutions [Itzhaki, Maldacena et al.]

Supersymmetry (SUSY) on the lattice

Beset by difficulties from the start because of SUSY algebra. The algebra is an extension of Poincare algebra by supercharges Q. Roughly, $\{Q,\overline{Q}\} \sim P$ and P generates infinitesimal translations which is broken on the lattice. SUSY algebra not satisfied at the classical level.

Solution:

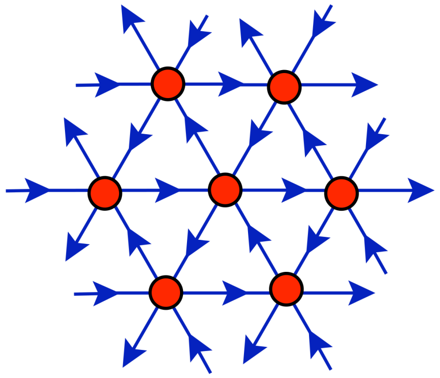
Preserve a subset of this algebra and hope that the supersymmetry is restored as continuum limit is taken. This idea has led to an improved understanding and has used for the results mentioned later in this talk. For review see: **0903.4881**

[Cohen, Kaplan, Katz, Unsal, Catterall, Sugino] $\rightarrow$ orbifolding/topological twist

Possible SYMs on the lattice

Theory	**R-symmetry group**	**Orbifolding**	**Maximal Twist**
$d=2, \mathcal{Q}=4, \mathcal{N}=2$	$SO(2) \bigotimes U(1)$	Yes	Yes
$d=2, \mathcal{Q}=8, \mathcal{N}=4$	$SO(4) \bigotimes SU(2)$	Yes	Yes
$d=2, \mathcal{Q}=16, \mathcal{N}=8$	$SO(8)$	Yes	Yes
$d=3, \mathcal{Q}=4, \mathcal{N}=1$	$U(1)$	No	No
$d=3, \mathcal{Q}=8, \mathcal{N}=2$	$SO(3) \bigotimes SU(2)$	Yes	Yes
$d=3, \mathcal{Q}=16, \mathcal{N}=4$	$SO(7)$	Yes	Yes
$d=4, \mathcal{Q}=4, \mathcal{N}=1$	$U(1)$	No	No
$d=4, \mathcal{Q}=8, \mathcal{N}=2$	$SO(2) \bigotimes SU(2)$	No	No
$d=4, \mathcal{Q}=16, \mathcal{N}=4$	$SO(6)$	Yes	Yes

Lattice $\mathcal{N} = 4$ SYM



This talk will present results based on the geometric construction and idea of twisting a supersymmetric gauge theory. This generates the 0-form supercharges needed to preserve a subset of SUSY algebra. In some sense, this is just a way of rewriting original fields and is justified for flat Euclidean space studies. Supercharges are broken into p-forms and then put on the lattice sites, links, plaquettes respectively.

- To reduce the fine-tuning to minimum and to identify the twisted fields in a consistent manner, we cannot just work with hypercubic lattice. We need what is called A_4^* lattice. Four-dimensional analog of $2d$ triangular lattice shown.

- S_5 point group symmetry and five links which is very natural to lay out fields of the $\mathcal{N} = 4$ SYM theory. But, basis vectors are not orthogonal.

Lower-dimensional SYM theories

As nice as it sounds, on the lattice $\mathcal{N} = 4$ SYM is extremely hard and not practical to study at large N and λ using classical computers. Also, there is a sign problem which we have observed for $\lambda \geq 5$. Additional complications because of being a super conformal field theory with no scales, moduli etc.

My research has focused over the years on the lower-dimensional version of this theory which are not conformal and the 't Hooft coupling is dimensional and computational costs are under control. Also, sign problem does not seem to play a role for range of couplings for interesting finite-temperature black hole Physics.

Matrix Models

Obtained by dimensional reduction of $\mathcal{N} = 1$ SYM from ten dimensions down to one. The theory has $SU(N)$ gauge symmetry and $SO(9)$ internal symmetry group corresponding to nine scalars. Simplest holographic gauge theory with well-defined gravity dual corresponding to Einstein's gravity.

$$S_{\mathrm{BFSS}} = \frac{N}{4\lambda} \int dt \mathrm{Tr} \left[(D_t X^i)^2 - \frac{1}{2} \left[X^I, X^J \right]^2 + \Psi^T D_t \Psi + i \Psi^T \gamma^j [\Psi, X^j] \right]$$

$$S_{\mathrm{BMN}} = S_{\mathrm{BFSS}} + S(\mu)$$

μ-terms break the $SO(9) \to SO(6) \otimes SO(3)$. BFSS has a single deconfined phase but BMN model admits a phase transition to confined phase!

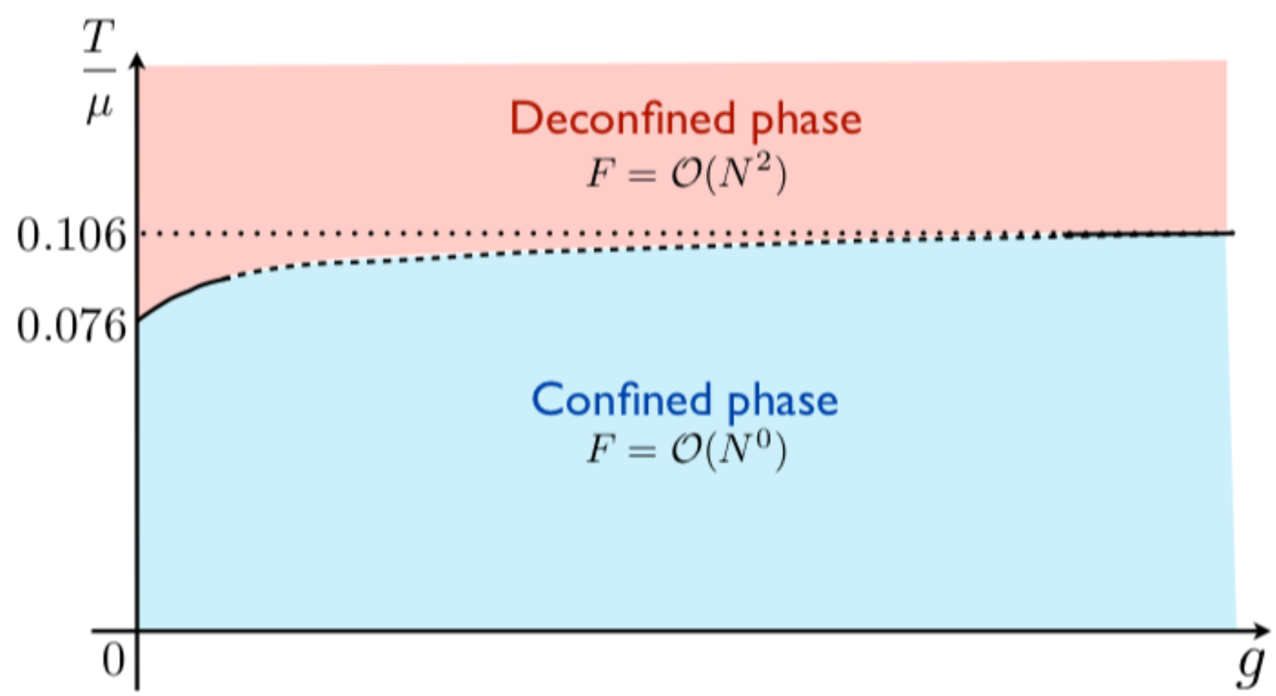
BFSS := Banks-Fischler-Shenker-Susskind

BMN := Berenstein-Maldacena-Nastase

BMN matrix model [2003.01298, 2105. XXXXX]

$$S_{BMN} = S_{BFSS} - \frac{N}{4\lambda}\int d\tau\ \mathrm{Tr}\left(\frac{\mu^2}{3^2}\left(X^I\right)^2 + \frac{\mu^2}{6^2}\left(X^M\right)^2 + \frac{2\mu}{3}\epsilon_{IJK}X^IX^JX^K + \frac{\mu}{4}\overline{\Psi}^\alpha\left(\gamma^{123}\right)_{\alpha\beta}\Psi^\beta\right)$$

The flat directions of the BFSS model are lifted by giving masses to SO(3) and SO(6) scalars. In addition, there is a cubic scalar term which is also known as 'Myers term' plus a fermion term. Dual gravity solution applicable when $g = \lambda/\mu^3 \gg 1$ with $\mu \ll 1, N \to \infty$.



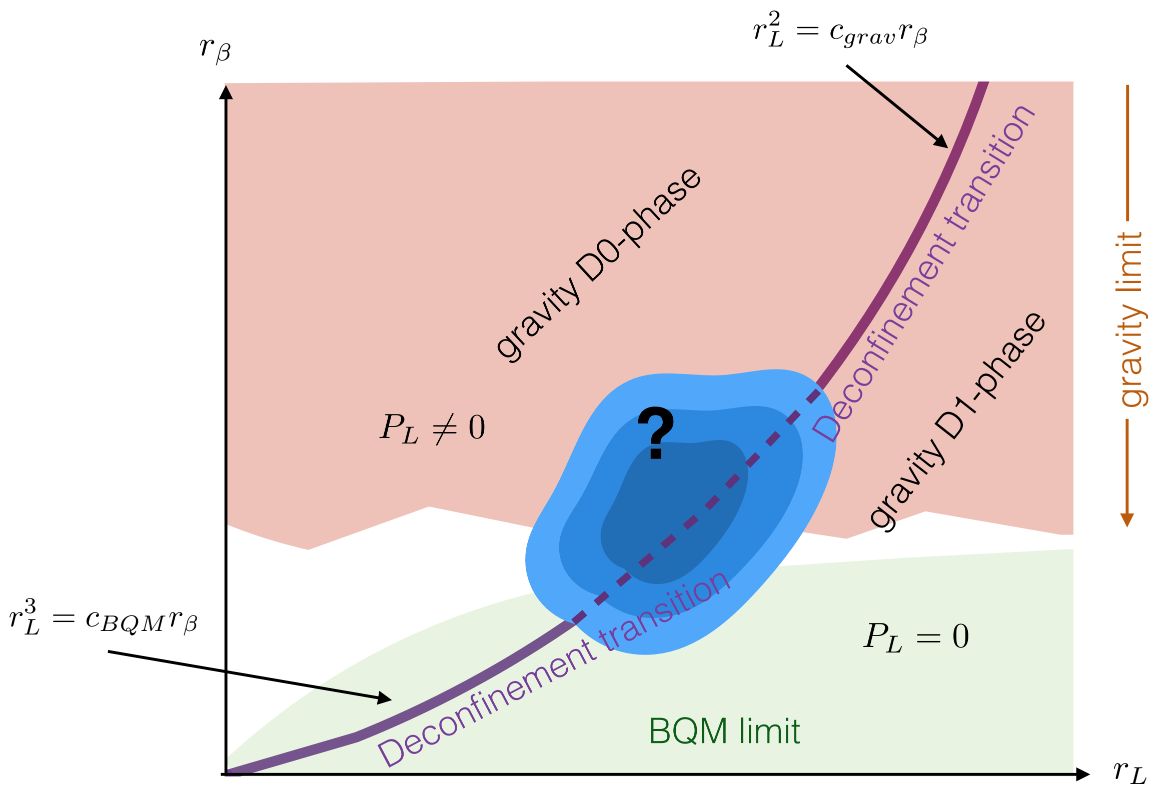
1411.5541 (Costa, Penedones, Greenspan, Santos)

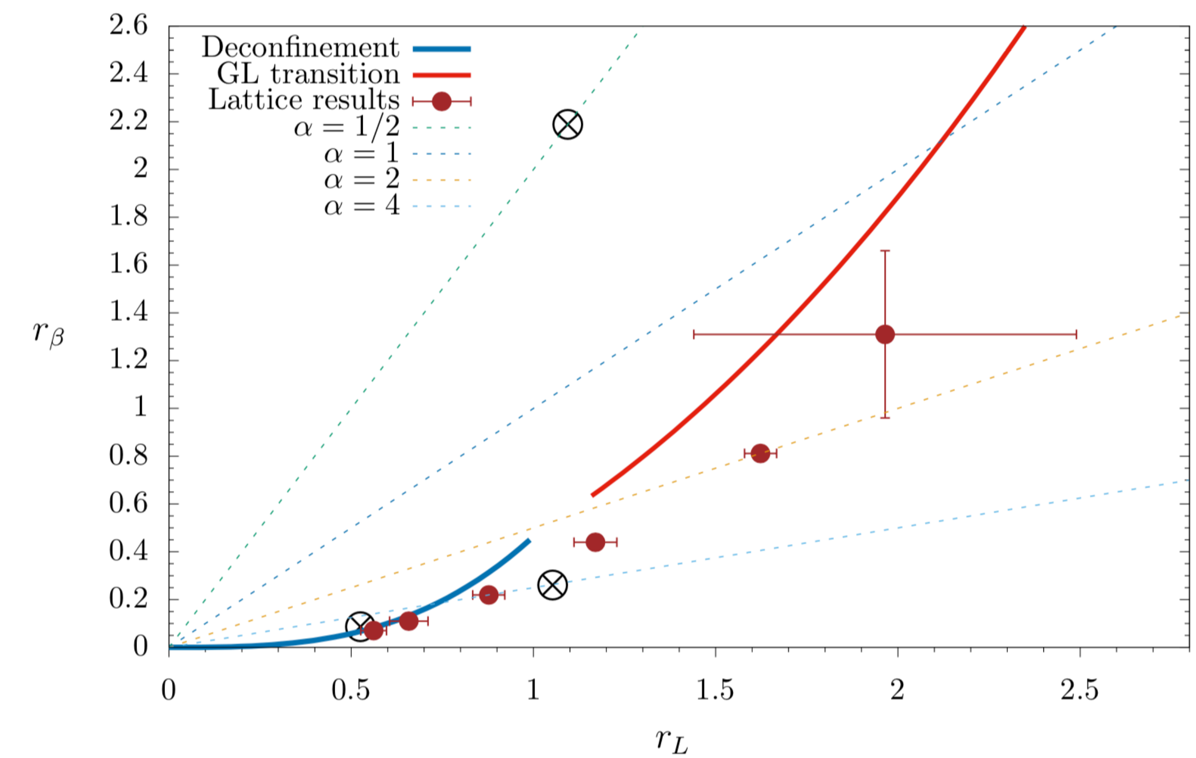
1+1-dimensional maximal SYM

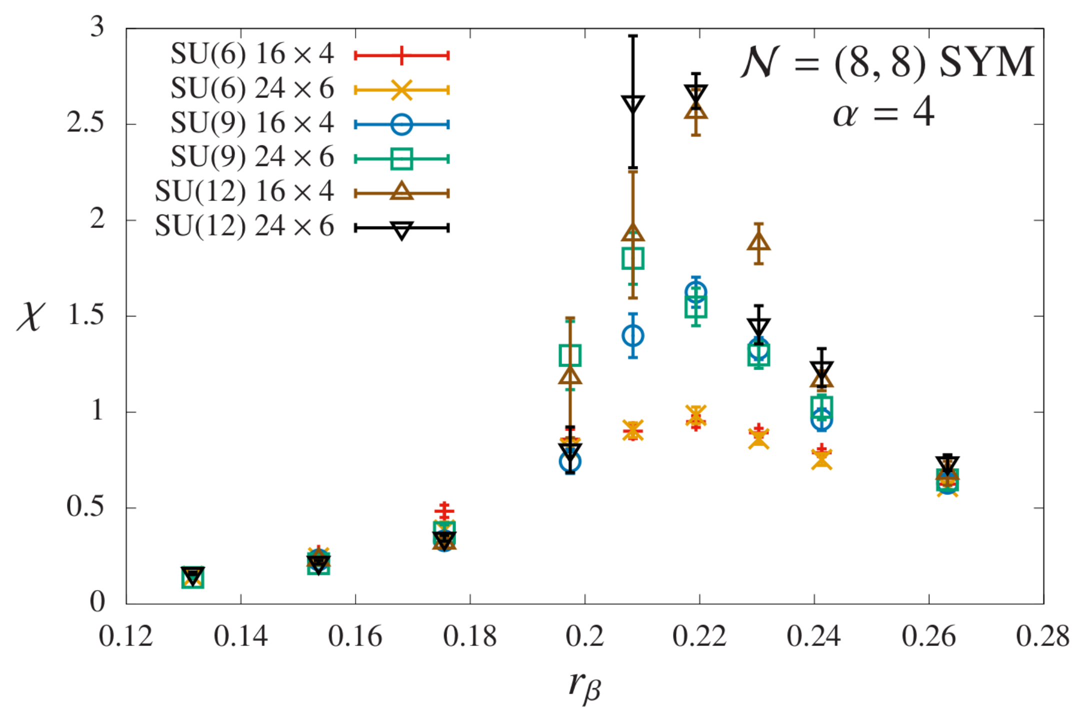
Dimensionally reduce the four-dimensional theory we have discretized on A4* lattice.

- Dimensionless couplings, $r_x = \lambda L$, $r_\beta = \lambda\beta$ and $\alpha = r_x / r_\beta$
- Study the deconfinement phase transition dual to topological transition between different supergravity black hole solutions. Expected at $\alpha^2 r_\beta \sim 2.45$ related to Gregory-Laflamme instability.
- Different phases have different parametric dependence on coupling/temperature.

Results from 1+1 SYM [published in PRD, 2018]







2+1-dimensional SYM

The three-dimensional SYM also has a holographic description at large N and strong coupling. The weak coupling (high-temperature) thermodynamic behaviour is just expected to be $\sim t^3$ while the power-law behaviour of the energy density is different at strong coupling. Contrast this with $\mathcal{N} = 4$ SYM which always has $\sim T^4$ dependence.

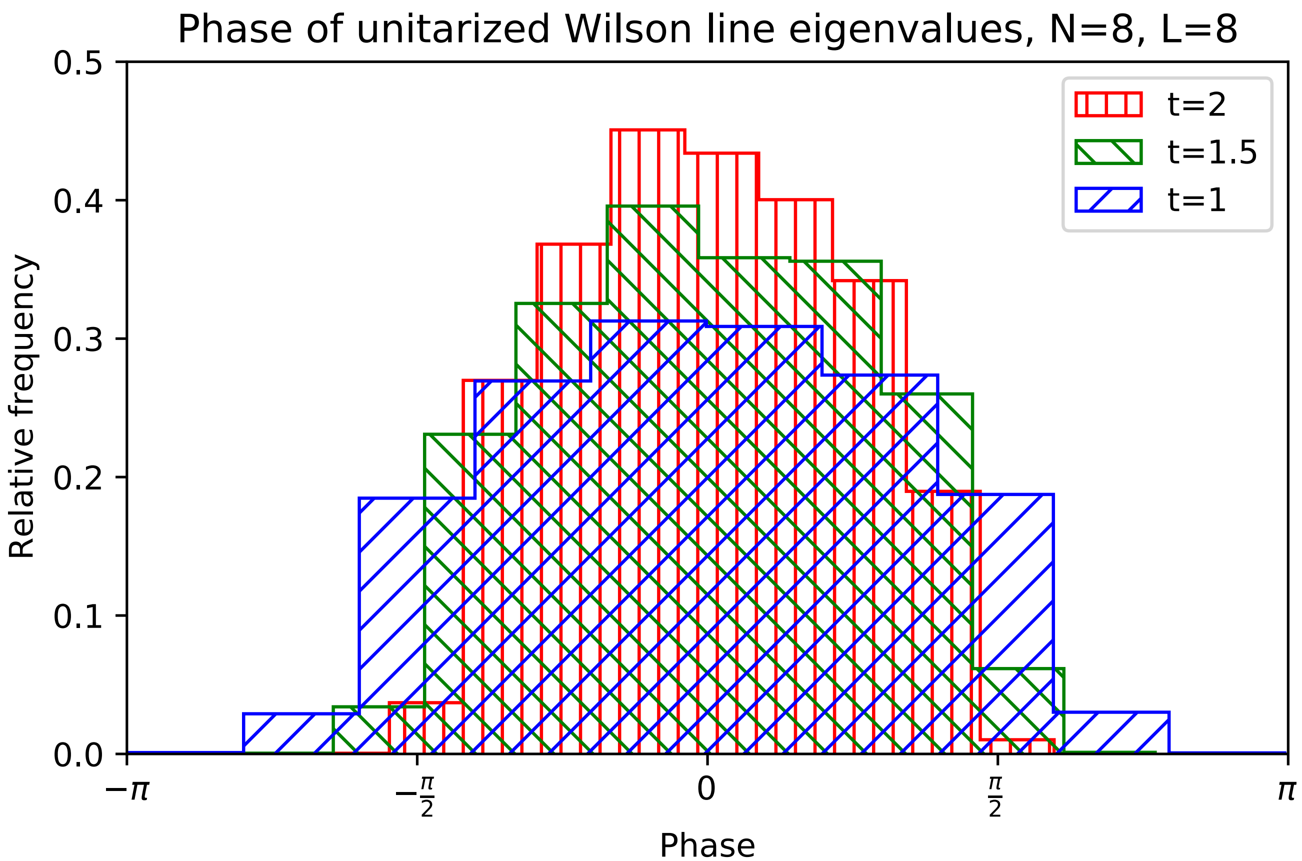
$$\frac{s_{\text{Bos}}}{N^2\lambda^3} = -\,0.831 t^{10/3} \qquad\qquad \frac{s_{\text{Bos}}}{N^2\lambda^3} = -\,2.598\ldots t^3$$

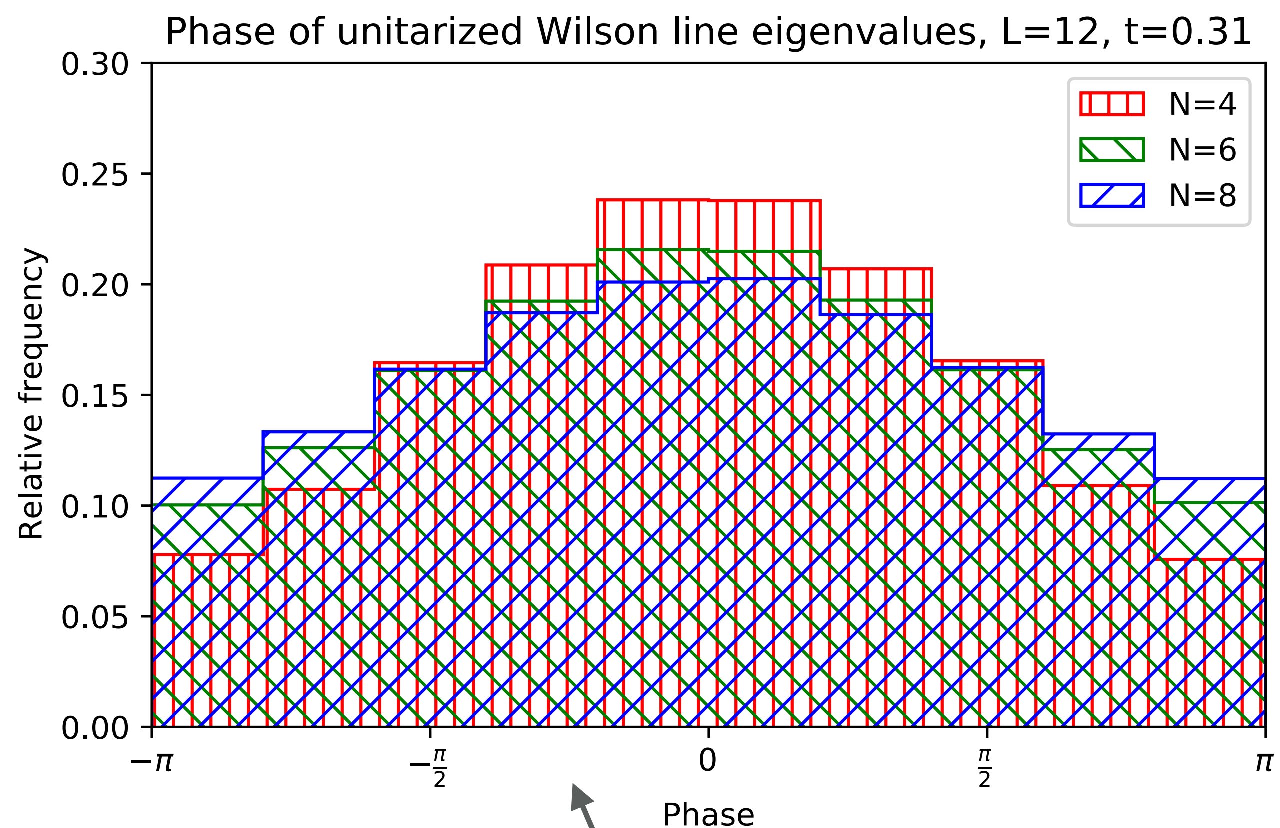
$t \sim 1$

increasing $t = T/\lambda$

It is worth noting that for SYM on an analogous torus in $p+1$-dimensions we would have parametric dependence $s_{\text{Bos}} \propto t^{(14-2p)/(5-p)}$ for $t \ll 1$ from the gravity dual, and the $t \gg 1$ limit would go as $s_{\text{Bos}} \propto t^{p+1}$. In the $p = 3$ conformal case these powers coincide

Results [published in PRD, 2020]



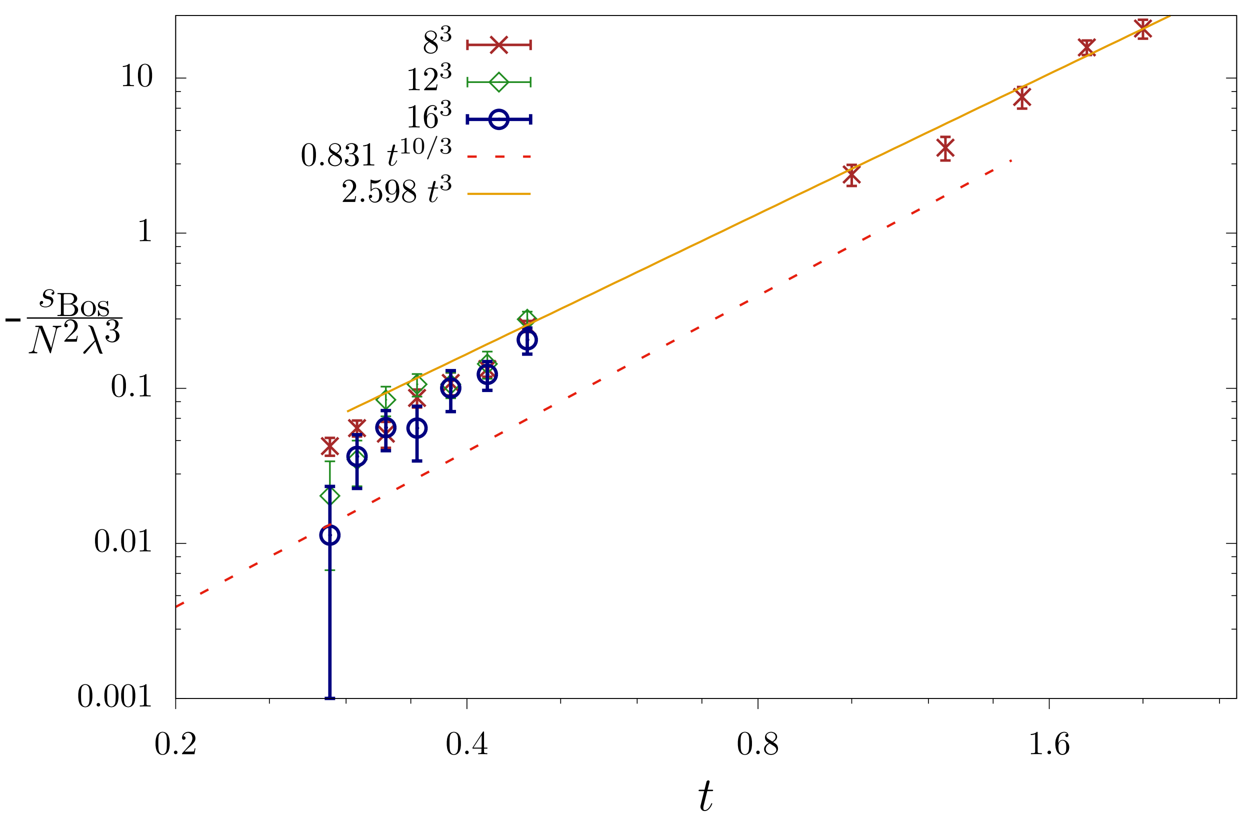


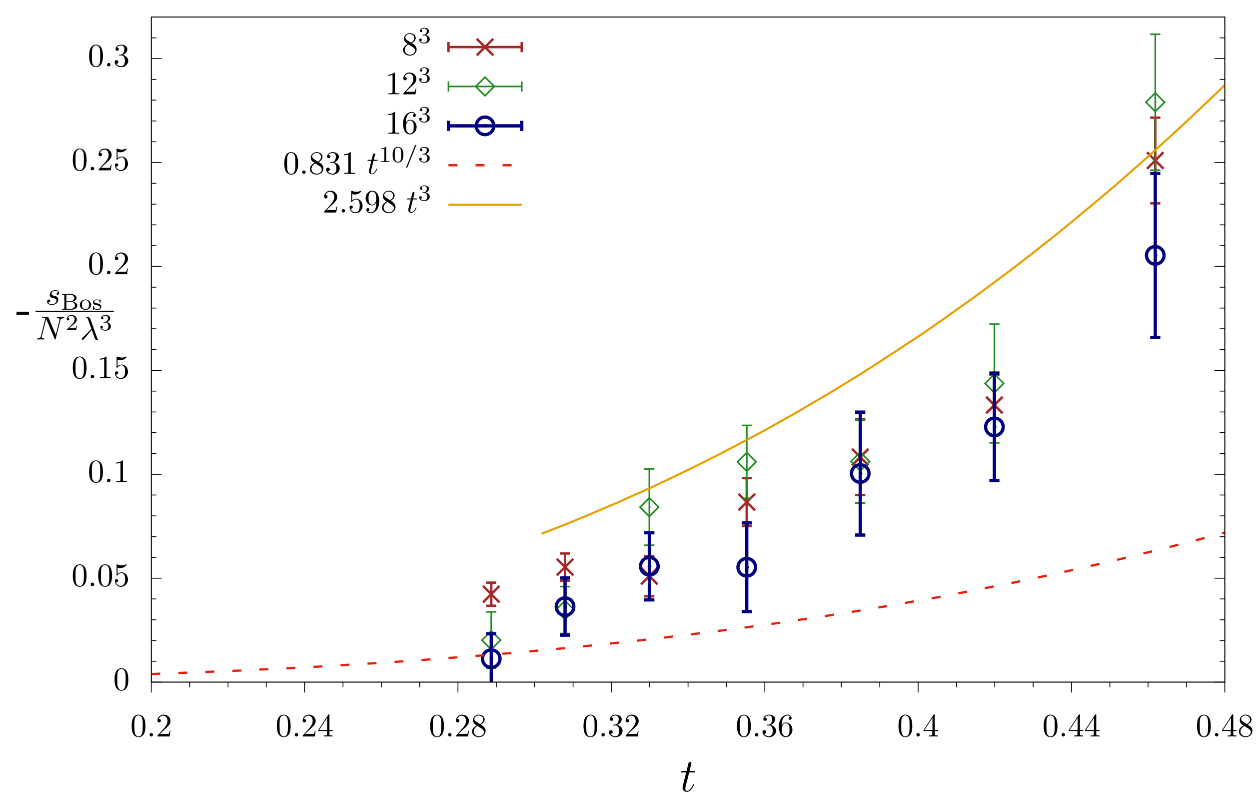
Uniform D2 phase.

Localized phase (high T)

Results [published in PRD, 2020]

We checked that high-T behaviour is reproduced by our lattice computations and that it agrees to supergravity results at low-temperatures.





Minimal SUSY & phenomenology

Holographic conjectures need maximal amount of SUSY (extended theories) like $\mathcal{N}=4$ SYM in $4d$. However, this theory cannot accommodate fermions transferring in the fundamental representation of the gauge group. We need to look beyond and to most notably $\mathcal{N}=1$ SYM in $4d$ which are arguably most important of all SUSY in $4d$ from point of view of extensions of SM.

If we dimensionally reduce this to two dimensions, we get $\mathcal{N}=(2,2)$ SYM. We studied this model few years back and looked for signals of dynamical SUSY breaking.

Results [published in PRD, 2018]

Witten index (I_W) can be useful (sometimes) to identify SUSY breaking defined as: $\mathrm{Tr}(-1)^{\mathrm{F}}$. If $I_W \neq 0$ then SUSY is unbroken while if $I_W = 0$, it may or may not be broken. But computing this directly using Monte Carlo simulations on the lattice is not possible since we can only compute expectation values. So, instead we looked at the vacuum energy in this model at finite temperatures and carefully took the zero temperature limit. We found that SUSY is unbroken in the continuum limit for $N = 2,3$. For this case, Hori & Tong in arXiv:0609032 had computed $I_W = 0$ and possibility of SUSY breaking was open. This is one example where lattice computation directly provided input to cross-check and improve continuum result.

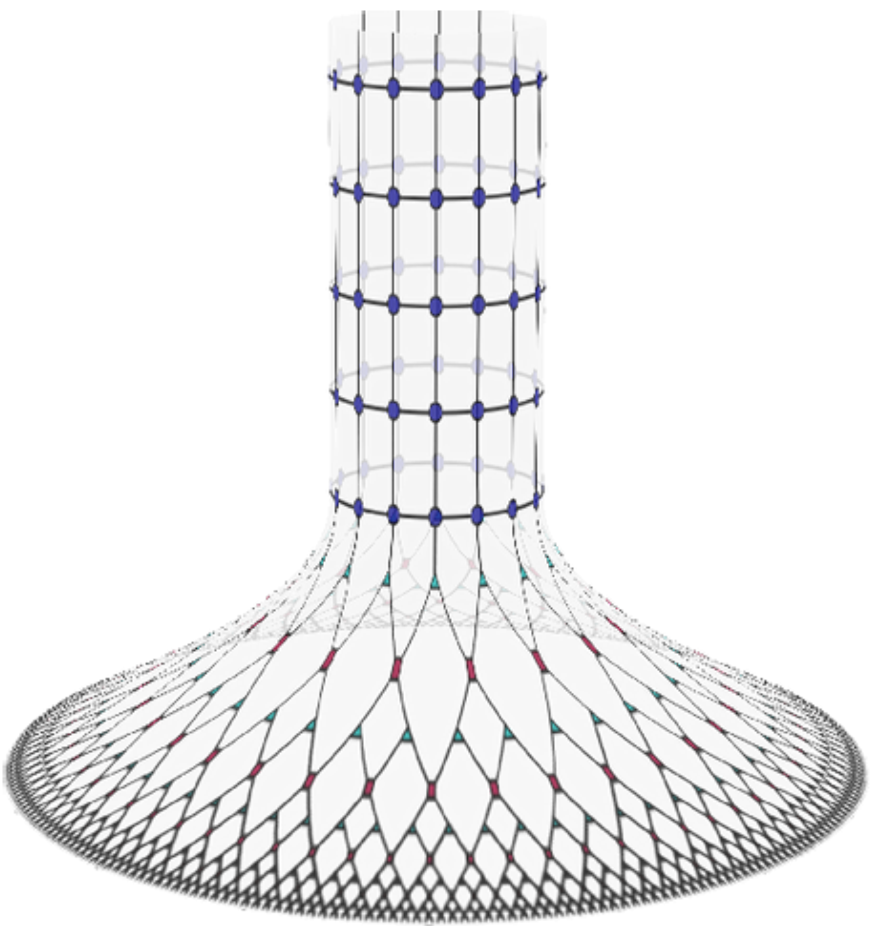
Also investigations by [Kanamori, Suzuki, Wipf et al.]

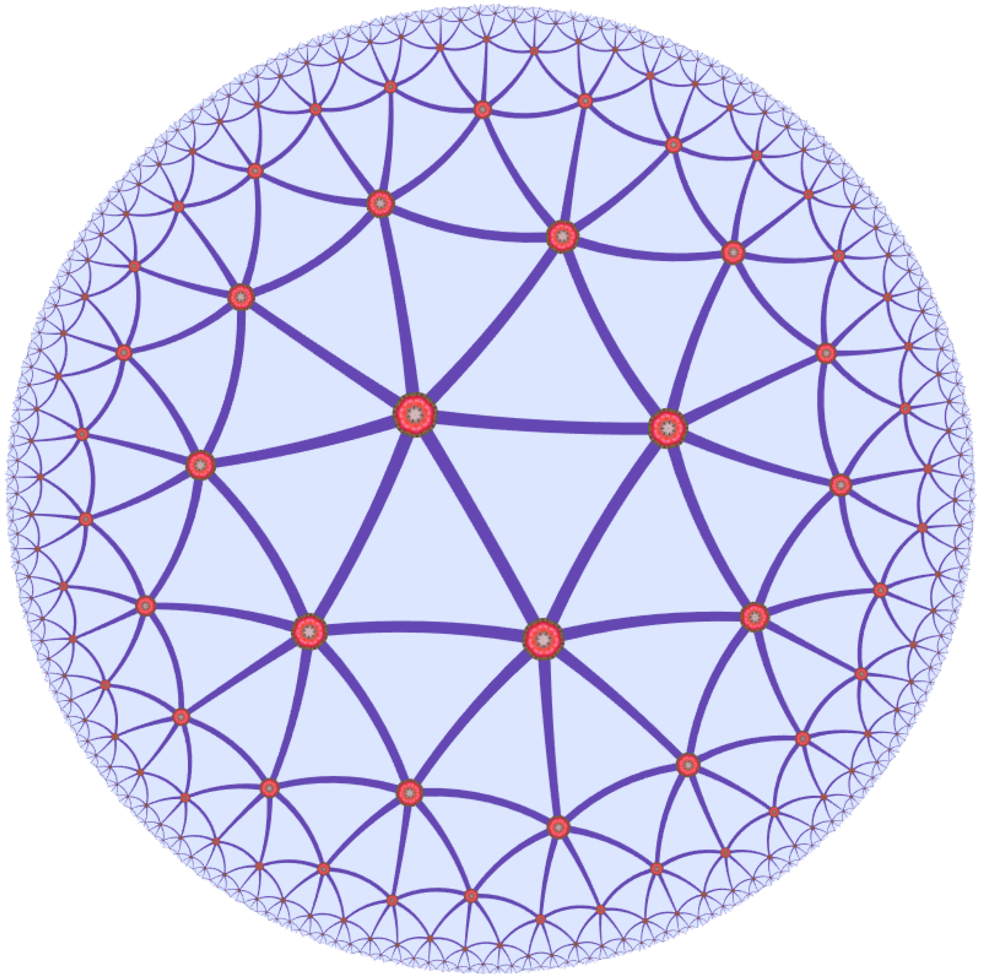
$\mathcal{N} = (2,2)$ SYM with matter

To study more interesting problems, one has to add matter (to make phenomenological connections) in fundamental/anti-fundamental representation of the gauge group. This has been studied for two-dimensional theory by several authors. The mass spectrum is interesting to study in the continuum limit. Once this is well-studied, the natural extension is to four-dimensional $\mathcal{N} = 1$ SYM where it is possible to study without exploiting any $\mathcal{Q}-$ exact techniques. This is because fine-tuning can be managed (only one relevant SUSY violating operator). This is work we want to pursue in the future in addition also 2d and 3d SQCD theories. However, we would need to use Ginsparg-Wilson fermions so that we maintain some discrete chiral symmetry on the lattice.

Looking ahead with SUSY!

- We want to study the Maldacena-Wilson loop and compare to holographic predictions.
- A big problem is to understand the coupling-dependence of free energy/entropy of $\mathcal{N} = 4$ SYM at finite-temperatures using lattice!
- Understand the static potential and anomalous dimensions of operators in $\mathcal{N} = 4$ SYM.





Tensor Network

Different renormalization group (RG) methods have been introduced over the past 5-6 decades:

- Kadanoff's spin blocking RG [1966] & Wilson's Numerical RG [1970s]

- Density Matrix Renormalization Group (DMRG) [White, 1992]
  (DMRG is a refined extension to above approach and is well-suited to all 1d systems not only restricted to impurity problems such as Kondo problem.)

- Tensor Renormalization Group [Levin and Nave, 2007]
  (More efficient than DMRG but breaks down at criticality as former.)

- Tensor Network Renormalization (TNR) [Vidal and Evenbly, 2015]
  (TNR is an extension of TRG which qualitatively improves TRG behaviour for systems at criticality and can be used to generate MERA tensor networks)

Motivations

- Formulating in terms of tensors can enable us to study systems where the usual Monte Carlo (MC) methods fail (sign problem!). In addition, the partition function is directly accessible in the thermodynamic limit unlike MC methods.

- Provides an arena for studying lower-dimensional critical and gapped systems faster and more efficiently than any other numerical method available today.

- Has recently been understood to play an important role in understanding the AdS/CFT (i.e. bulk physics from entangled quantum state at the boundary).

Basic idea

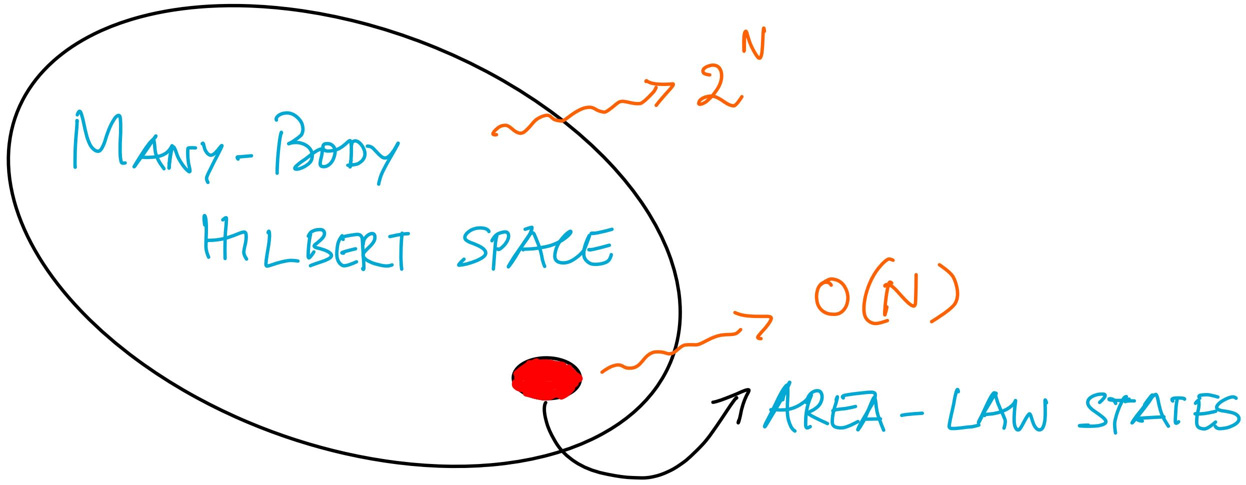
Tensor networks approach belong to two categories: Lagrangian and Hamiltonian approaches. For ex:

$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} C_{i_1 \cdots i_N} | i_1 i_2 \cdots i_N \rangle$ as approximation to the ground state wave function of complicated many-body quantum system with local Hamiltonian. Ex: Matrix Product States (MPS) representation. Reduction to $O(N)$ rather than $O(d^N)$ coefficients.

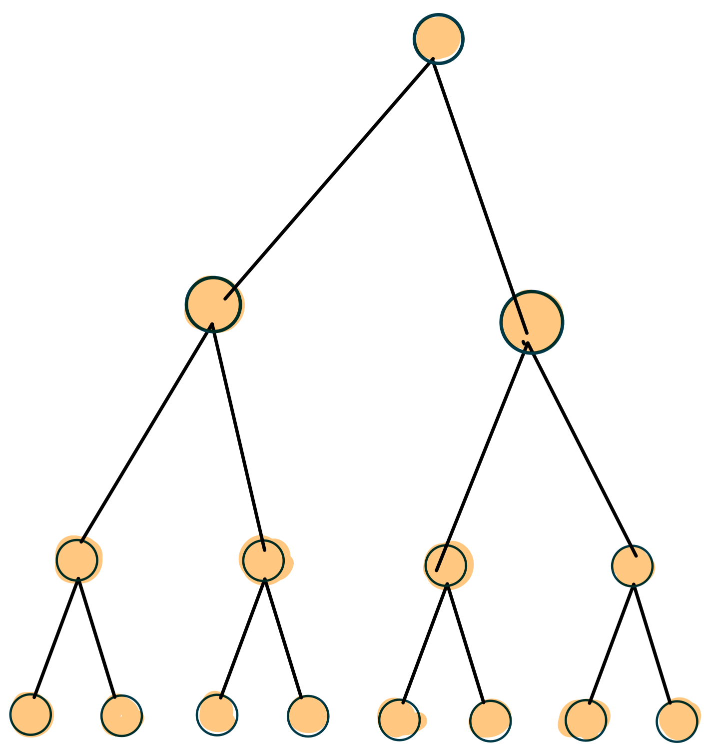
$Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})}$ to approximate in the Lagrangian formulation (like we consider in this talk later).

How to identify relevant states?

- Ground states are not "arbitrary" states in the Hilbert space, it has some special features. These have been captured by studying the entanglement entropy (EE). The region of Hilbert space that obeys area-law scaling for the EE corresponds to a tiny corner (in red). Therefore, lot of progress have been made in many-body physics by computing EE and hence identifying important regions of Hilbert space.



TTN/MERA (Multi-scale Entanglement Ansatz)



TREE TENSOR NETWORK

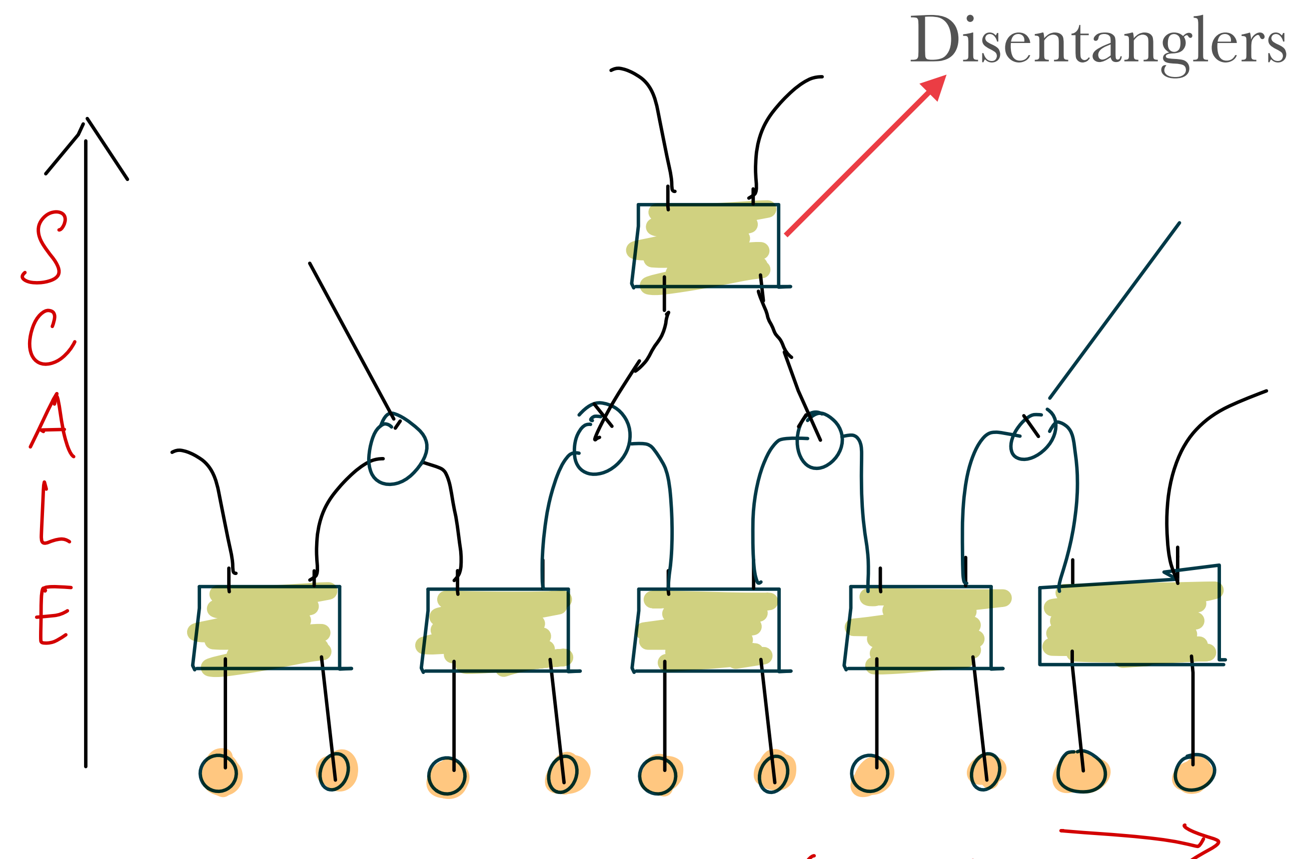
(gapped systems)

↓

finite correlation length

↓

Area-law

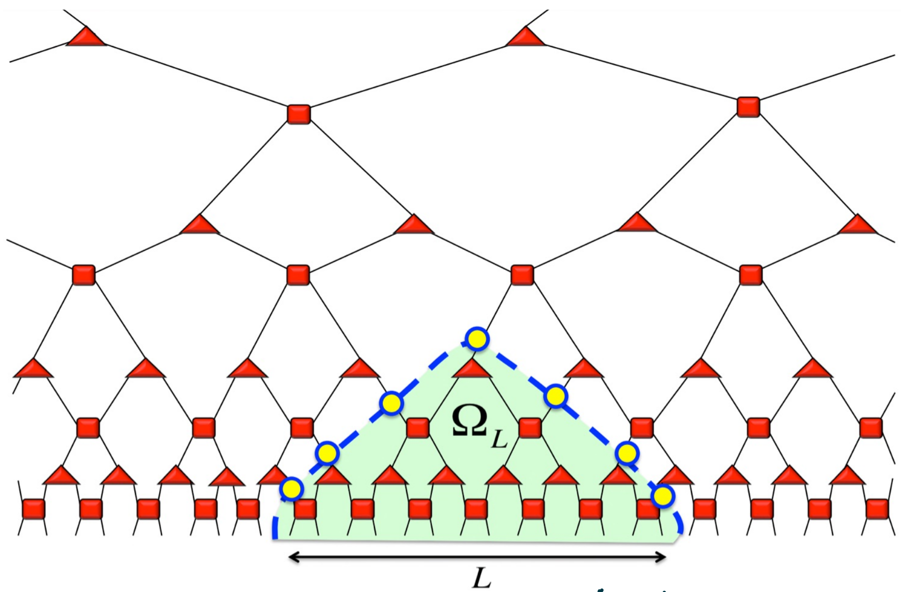


MERA (1-d) L

(effecient for critical system)

S_A has logarithmic behaviour

MERA



arxiv 1812.04011

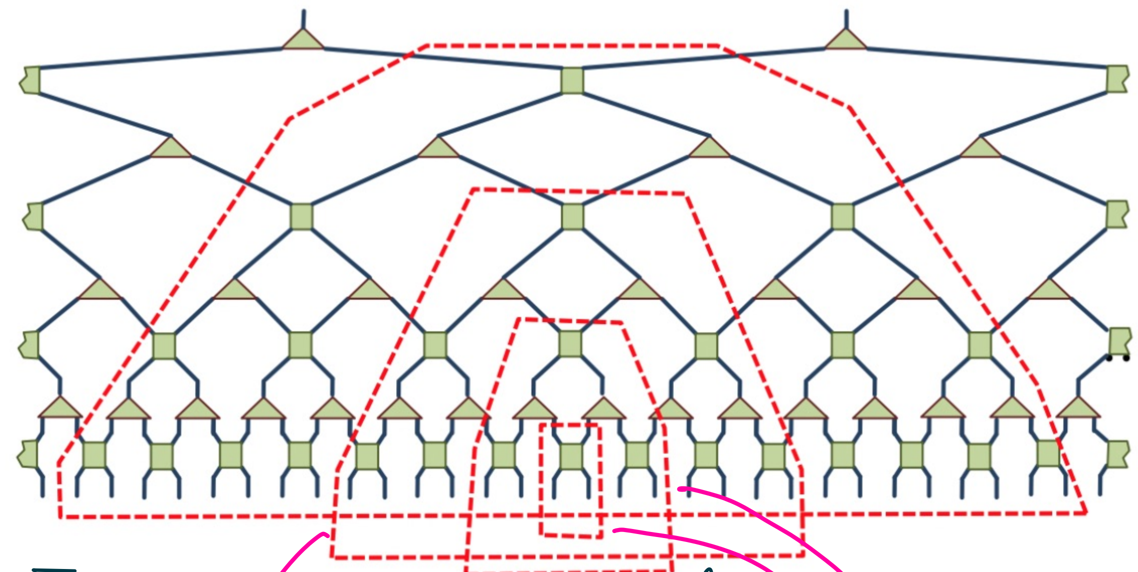


Figure Courtesy: G. Vidal

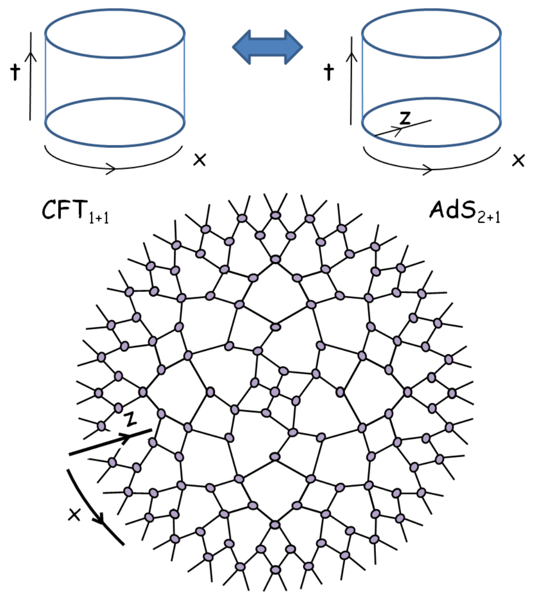
$\{14, 6\}$ $\{2, 2\}$ $\{6, 4\}$

$$\boxed{\partial\Omega_L = O(\log L)}$$

disentanglers: removing short-range entanglement $V \otimes V \to V \otimes V$

isometries: mapping block of sites to one $V \otimes V \to V$

What does MERA capture?



- Swingle (0905.1317) — MERA describes a time slice of the Poincare patch of AdS_3, which corresponds to the hyperbolic plane. So, MERA is the lattice realisation of AdS/CFT! (* Hint from Ryu-Takayanagi (RT) formula matching EE*)

- Beny and others (1110.4872) — MERA on the real line should be interpreted instead as a Poincare patch of dS spacetime

- Vidal & Milsted (1812.00529) argued that MERA on the real line would describe light sheet geometry.

HOTRG (Higher-order TRG)

A refined real space coarse graining method similar in spirit to TRG but employs higher-order SVD (HOSVD) to minimise the errors due to truncation. First introduced in 1201.1144 and is successfully applied to statistical systems in various *d*. Performs better than naive TRG for critical systems. Less complex than the TNR methods (best for critical systems!)

Coarse-graining renormalization by higher-order singular value decomposition

Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, T. Xiang

We propose a novel coarse graining tensor renormalization group method based on the higher-order singular value decomposition. This method provides an accurate but low computational cost technique for studying both classical and quantum lattice models in two- or three-dimensions. We have demonstrated this method using the Ising model on the square and cubic lattices. By keeping up to 16 bond basis states, we obtain by far the most accurate numerical renormalization group results for the 3D Ising model. We have also applied the method to study the ground state as well as finite temperature properties for the two-dimensional quantum transverse Ising model and obtain the results which are consistent with published data.

Example - 2d Ising [Square lattice]

Exactly solvable system with solution due to Onsager, where the logarithm of the partition function is given by:

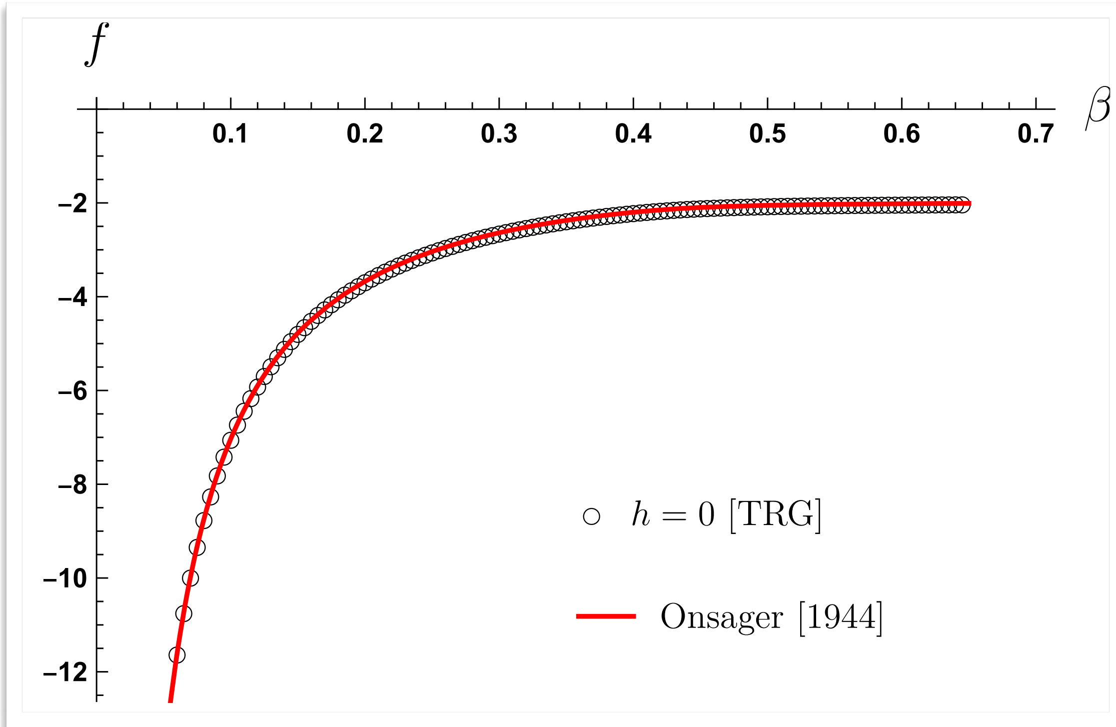
$$f(\beta) = -\frac{1}{\beta}\left(\ln(2) + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln\left[2\cosh^2(2\beta) - \sinh(2\beta)\cos(\phi_1) - \sinh(2\beta)\cos(\phi_2)\right] d\phi_1 d\phi_2 \right)$$

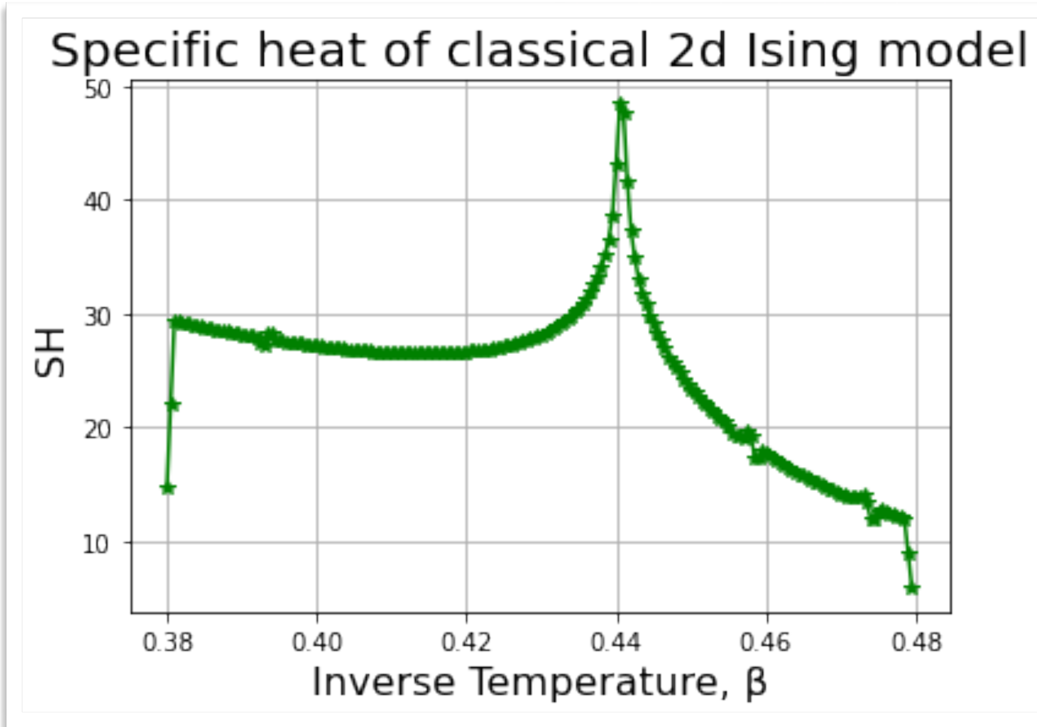
And has singularity (phase transition) at:

$$T_c = \frac{2}{\ln(1+\sqrt{2})} = 2.26918531421 \implies \beta_c \approx 0.440687$$

We apply HOTRG to this system and match to known analytical results as a check to show its effectiveness.

State-of-the-art numerical result





Time: 100 seconds on a modern laptop for this plot.

Classical XY model

[Journal of Stat. Mech, 2020], **2004.06314**

Simplest spin model with continuous symmetry $O(2)$ in two dimensions. The nearest neighbour Hamiltonian is given by:

$$\mathscr{H} = -J\sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) - h\sum_i \cos\theta_i$$

In order to construct the tensor representation, we decompose the Boltzmann weight (for say $h = 0$) using Jacobi-Anger expansion as:

$$\exp\left(\beta\cos(\theta_i - \theta_j)\right) = I_0(\beta) + \sum_{\nu=-\infty,\neq 0}^{\infty} I_\nu(\beta) e^{i\nu(\theta_i - \theta_j)}$$

where, I_ν is the modified Bessel function of first kind.

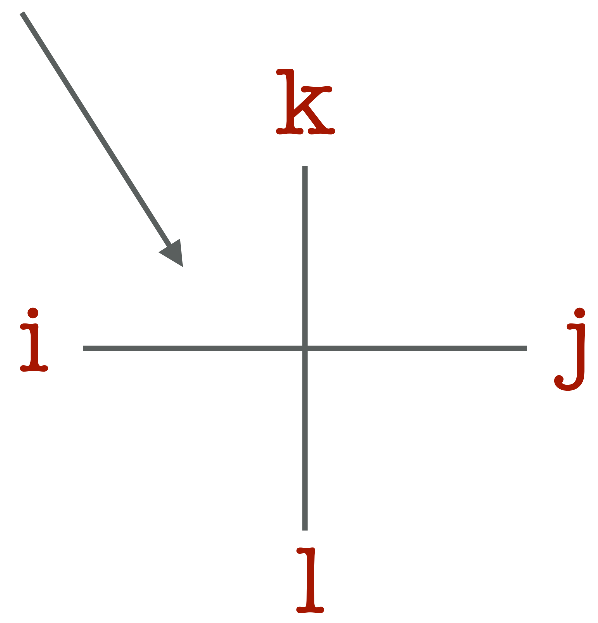
Classical XY model

The partition function can then be written as:

$$Z = \int \prod_i d\theta_i \prod_{\nu_{ij},\mu_i} I_{\nu_{ij}}(\beta) I_{\mu_i}(\beta h) e^{i\nu_{ij}(\theta_i - \theta_j) + i\mu_i\theta_i}$$

By integrating over $d\theta_i$, we obtain the initial tensor for XY model

$$T_{ijkl} = \sqrt{I_i(\beta) I_j(\beta) I_k(\beta) I_l(\beta)}\, I_{i+k-j-l}(\beta h)$$



Taking the limit $h \to 0$, we obtain the critical temperature of 0.89290(5) which is consistent with the most precise MC results available in literature. This result can be further improved using larger bond dimension χ. It is known that in XY model correlation length increases exponentially as we approach $T = T_c$. Already at $T \approx 0.95$, the correlation length as noted in 1907.04576 is more than thousand lattice sites. Precise MC results used a square lattice of size $2^{16} \times 2^{16}$ while we used a lattice $2^{50} \times 2^{50}$.

METHOD	YEAR	SYSTEM SIZE	T_{critical}	
Monte Carlo [21]	1992	$2^9 \times 2^9$	0.89400(500)	
HTE [22]	1993	–	0.89440(250)	
Monte Carlo [23]	1995	$2^8 \times 2^8$	0.89213(10)	
Monte Carlo [24]	2005	$2^{11} \times 2^{11}$	0.89294(8)	
HTE [25]	2011	–	0.89286(8)	
Monte Carlo [15]	2012	$2^{16} \times 2^{16}$	0.89289(5)	
Monte Carlo [26]	2013	$2^9 \times 2^9$	0.89350(10)	
Higher-order TRG [7]	2013	$2^{40} \times 2^{40}$	0.89210(190)	TN
Uniform MPS [8]	2019	–	0.89300(10)	TN
Higher-order TRG [This work]	2020	$2^{50} \times 2^{50}$	0.89290(5)	TN

2d non-Abelian gauge Higgs (NAGH) model

[PRD, 2018], **1901.11443**

The lattice action for $SU(2)$ is given by:

$$S = -\frac{\beta}{2}\text{Re Tr}\square - \frac{\kappa}{2}\text{ Re TrU}$$

where β is the gauge coupling and κ is the matter coupling. We fix to unitary gauge as used in several previous works by Greensite et al. [Also Osterwalder-Seiler-Fradkin-Shenker].

We expand the Boltzmann weights in terms of characters (called character expansion). Writing, $S = S_g + S_\kappa$, we have for gauge piece:

$$e^{-S_g} = \prod_x \sum_r F_r(\beta)\chi^r(UUU^\dagger U^\dagger)$$

and similarly for the other term.

Link (A) and Plaquette (B) tensors

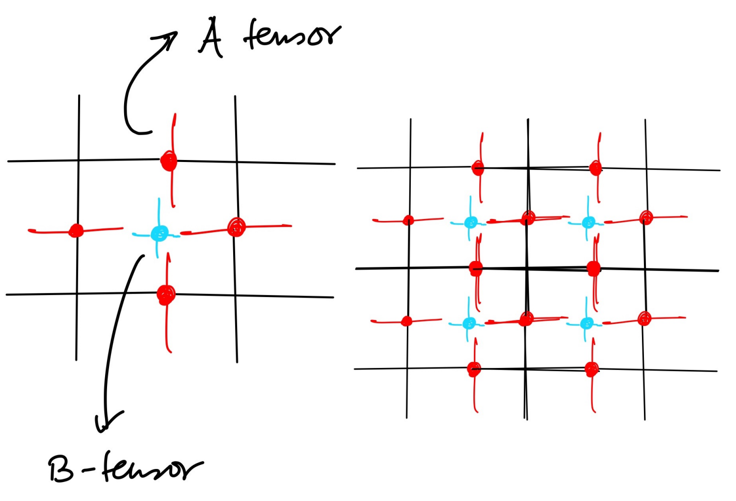
1901.11443

$$A_{(r_l m_{la} m_{lb})(r_r m_{ra} m_{rb})}(\kappa) = \frac{1}{d_{r_r}} \sum_{\sigma=|r_r-r_l|}^{r_r+r_l} F_\sigma(\kappa) C^{r_r m_{rb}}_{r_l m_{lb} \sigma(m_{rb}-m_{lb})} \times C^{r_r m_{ra}}_{r_l m_{la} \sigma(m_{rb}-m_{lb})}.$$

$$B_{(r_l m_{la} m_{lb})(r_r m_{ra} m_{rb})(r_a m_{al} m_{ar})(r_b m_{bl} m_{br})} = \begin{cases} F_r(\beta) \quad \delta_{m_{la}, m_{al}} \delta_{m_{ar}, m_{ra}} \delta_{m_{rb}, m_{br}} \delta_{m_{bl}, m_{lb}} & \text{if } r_l = r_r = r_a = r_b = r \\ 0 \qquad \text{else.} \end{cases}$$

With the knowledge of these two tensors, one can construct the fundamental tensor.

$$T_{ijkl}(\beta, \kappa) = \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta\gamma\delta}(\beta) L_{\alpha i} L_{\beta j} L_{\gamma k} L_{\delta l}(\kappa), \quad \text{where} \quad A_{ij} = \sum_k L_{ik} L^T_{kj}$$

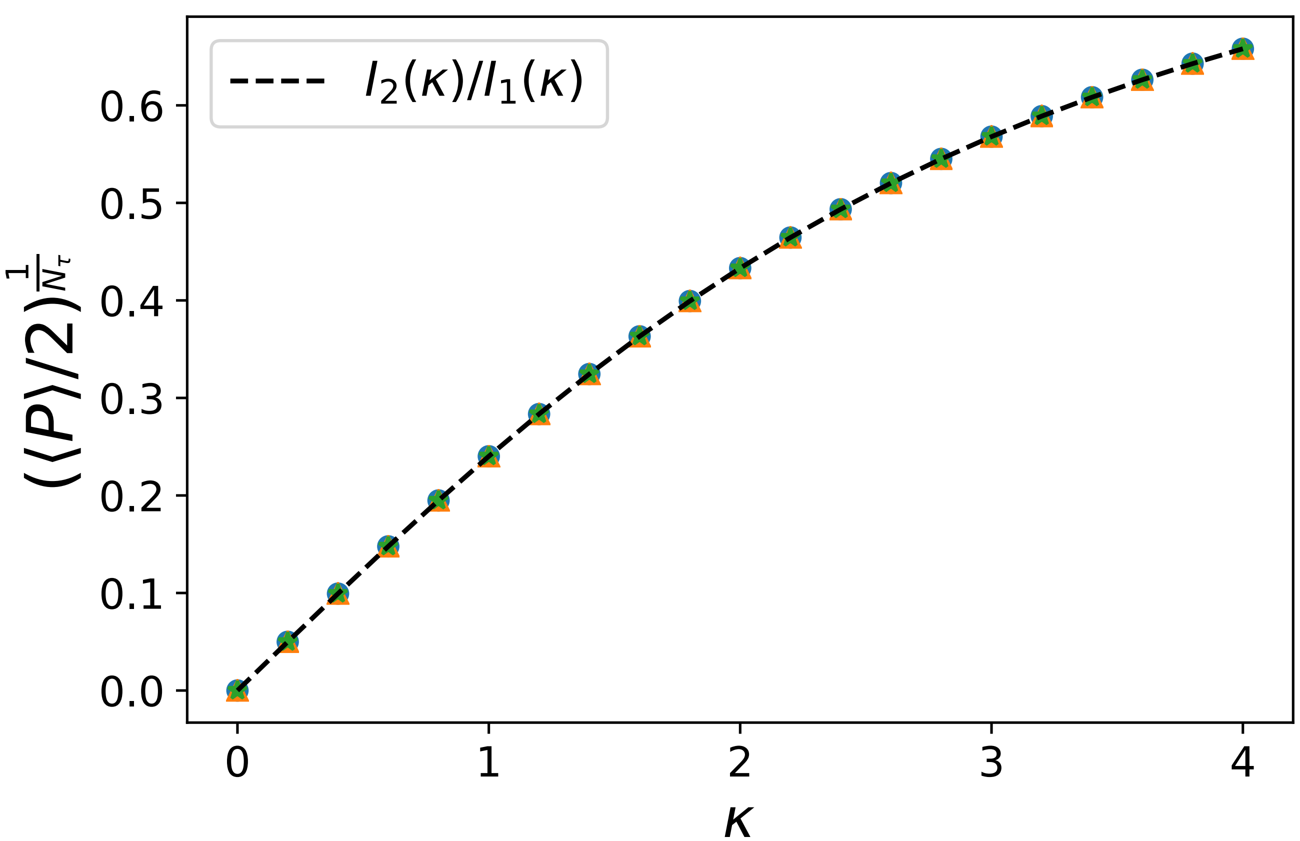


Exact results for $\beta = 0$

[1901.11443]

For $\beta = 0$, we can write exact value of the Polyakov loop in terms of Bessel functions. This provides a simple check of the tensor formulation. The exact expression is:

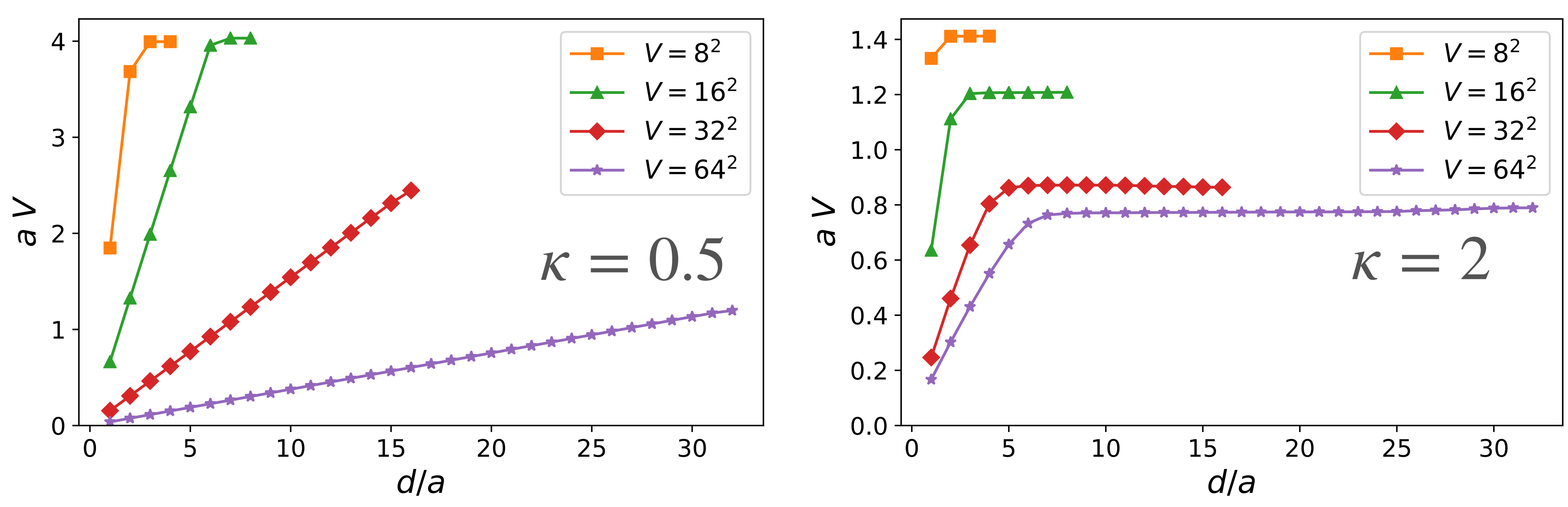
$$\langle P \rangle = 2 \left(\frac{I_2(\kappa)}{I_1(\kappa)} \right)^{N_\tau}$$



Results

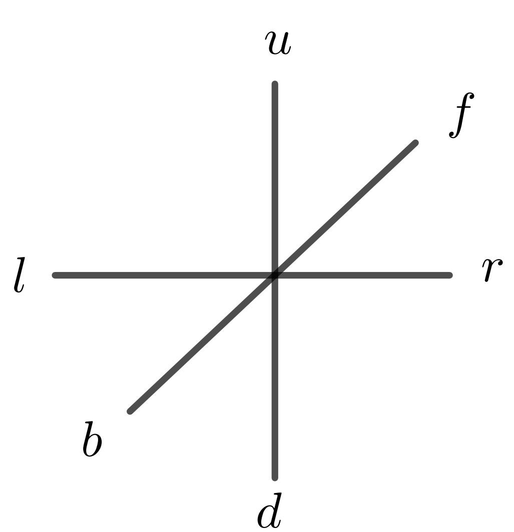
Polyakov loop correlator is given by, $C(d) = \exp(-\beta V(d))$. This also provides a measure of monitoring confinement when $V \propto d$ (the slope gives σ, string tension). In a Higgs phase, it is constant and independent of d.

We show the results for correlator in two phases separated by cross-over at around $\kappa \approx 1.4$. The left shows the confining phase while for the one on the right a string breaking occurs and it goes to Higgs phase for sufficiently large lattices. Results are consistent with Monte Carlo results from 1402.7124

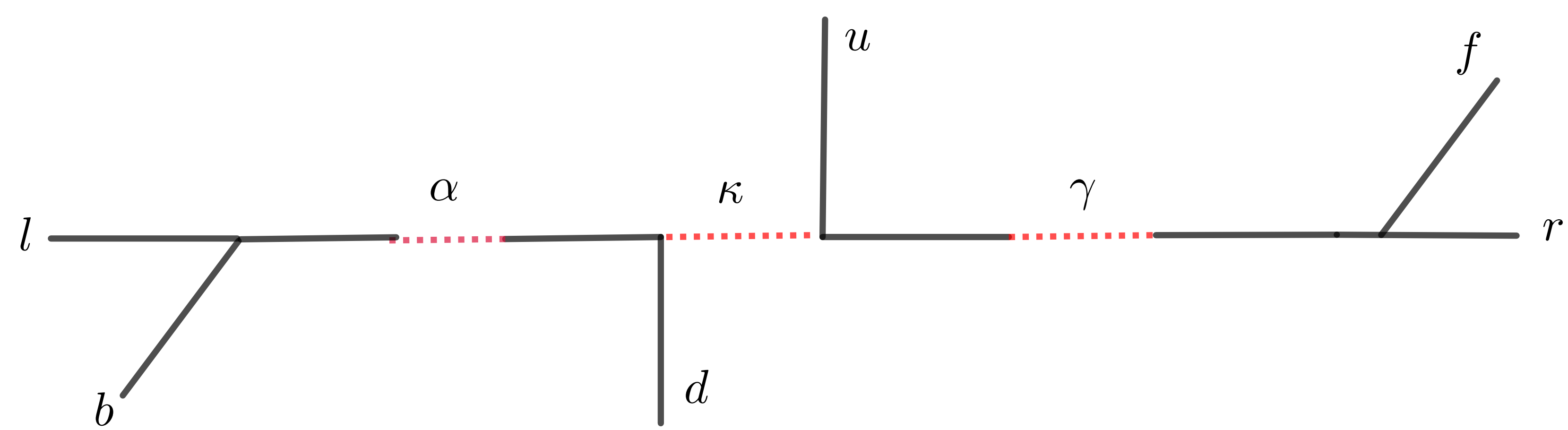


Current work in progress!

- We are currently exploring the tensor network methods for three-dimensional classical statistical systems. This has not been much pursued due to the computational complexity. However, due to some recent efficient representations of higher-dimensional tensors, this might be possible.



T_{lrudfb}



$$T_{lrudfb} = A_{lb\alpha} B_{\alpha d\kappa} C_{\kappa u\gamma} D_{\gamma fb}$$

Open problems!

- Study $SU(3)$ gauge theories using tensor networks & large N theories in the next decade or so ?
- Quantum simulations of spin models and study real-time evolution of supersymmetric lattice theories starting with $0+1$ dimensions starting with $SU(2)$ and then move to large N ?
- Develop higher-dimensional (i.e. $2+1$ or $3+0$) methods shed some light on connection to holography as MERA did?

Thank you