Holographic gauge theories on the lattice

Based on 2003.01298, 2010.00026 & work in progress

DIAS, 23 June 2021 [Online via Zoom]

Raghav Govind Jha

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Outline

- Introduction to holography (gauge/gravity)
- Supersymmetric theories on the lattice (starting with $\mathcal{N} = 4$ SYM in four dimensions)
- 3d $\mathcal{N} = 8$ SYM in the large N limit at finite temperatures
- Phase structure and thermodynamics of BMN matrix model
- Future directions & open problems

Holography

Basic idea: A quantum-gravitational theory in $d+1$ dimensions is related to some quantum field theory (without gravity) in one fewer dimension on its boundary.

It is now widely believed that any consistent theory of quantum gravity will be holographic. First hints came in 1970s, when Hawking & Bekenstein found that the black hole entropy was proportional to area rather than the volume as expected.

AdS/CFT conjecture

A well-defined correspondence was conjectured between a five-dimensional quantum theory of gravity in Anti- de Sitter (AdS) space-time and four-dimensional super-conformal field theory (CFT) on the boundary. In the decoupling limit of $N \to \infty, \lambda \gg 1$, the quantum gravity reduces to Einstein-like gravity in the bulk.

Beyond AdS/CFT

But, there is nothing special about the fundamental idea of holography to the pair of 4 & 5 dimensions. Soon after, it was defined for maximal supersymmetric gauge theories for $d < 4$ even though they are not conformal.

Statement: Maximally supersymmetric Yang-Mills (SYM) theory in p+1-dimensions is dual to Dp-branes in supergravity at low temperatures in a special limit (large N, strong coupling). In other words, the supergravity solutions corresponding to p+1 SYM are black p-brane solutions. In this talk we will talk about $p \leq 3$ only.

[Itzhaki et al. PRD **58**, 046004]

Class of SYM theories

All the supersymmetric theories which are known to have a well-defined holographic dual description descend from the same ten-dimensional $\mathcal{N} = 1$ SYM theory. Non-maximal SUSY theories do not admit holographic dual.

$$S = \int d^{10}x \, \mathrm{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} D_\mu \gamma^\mu \psi \right)$$

16 supercharges:

D=10, $\mathcal{N} = 1$ $\rightarrow$ D=6, $\mathcal{N} = 2$ $\rightarrow$ D=4, $\mathcal{N} = 4$ $\rightarrow$ D=3, $\mathcal{N} = 8$ $\rightarrow$ D=2, $\mathcal{N} = (8, 8)$

8 supercharges:

D=6, $\mathcal{N} = 1$ $\rightarrow$ D=4, $\mathcal{N} = 2$ $\rightarrow$ D=3, $\mathcal{N} = 4$

4 supercharges:

D=4, $\mathcal{N} = 1$ $\rightarrow$ D=3, $\mathcal{N} = 2$

Strong/Weak duality

Since the gauge/gravity is a strong/weak duality it is often not possible to compute on both sides simultaneously. This opens up the possibility of exploring the strongly coupled field theory using well-known *lattice* methods and compare to results obtained from weakly coupled quantum gravity theory. This is a non-trivial check of the validity of the duality. However, the gauge theories are complicated to numerical study on the lattice because of extended supersymmetry and requirement of large N limit.

$\mathcal{N} = 4$ SYM

Obtained by dimensionally reducing the ten-dimensional SYM theory down to four dimensions. It is a conformal field theory, β-function vanishes to all orders, consists of six scalars, sixteen real fermions, all massless and in the adjoint representation of the SU(N) gauge group. Simplest interacting QFT in four dimensions.

The action consists of kinetic, Yukawa, quartic scalar commutator terms and are all related by supersymmetry. The *superconformal* algebra includes an SU(4) = Spin(6) symmetry and is part of R-symmetry group apart from the usual SO(4) Euclidean group. At finite temperatures, SUSY is broken and in that limit sometimes dubbed as close cousin of QCD even though it is not physical though!

Supersymmetry on the lattice

Beset by difficulties from the start because of SUSY algebra. The algebra is an extension of Poincare algebra by supercharges Q and $\overline{Q}$. Roughly, $\{Q, \overline{Q}\} \sim P_\mu$ and P_μ generates infinitesimal translations which is broken on the lattice. SUSY algebra not satisfied at the classical level.

Alternative:

Preserve a subset of this algebra and check (expect!) that the supersymmetry is restored as continuum limit is taken. This idea has led to an improved understanding and has used for the results mentioned later in this talk. For review see: 0903.4881

[Cohen, Kaplan, Katz, Unsal, Catterall, Sugino]

during 2000-2010 using different but equivalent approaches.

Possible SYMs on the lattice

Theory	R-symmetry group	Orbifolding	Maximal Twist
$d=2, \mathcal{Q}=4, \mathcal{N}=2$	$SO(2) \bigotimes U(1)$	Yes	Yes
$d=2, \mathcal{Q}=8, \mathcal{N}=4$	$SO(4) \bigotimes SU(2)$	Yes	Yes
$d=2, \mathcal{Q}=16, \mathcal{N}=8$	$SO(8)$	Yes	Yes
$d=3, \mathcal{Q}=4, \mathcal{N}=1$	$U(1)$	No	No
$d=3, \mathcal{Q}=8, \mathcal{N}=2$	$SO(3) \bigotimes SU(2)$	Yes	Yes
$d=3, \mathcal{Q}=16, \mathcal{N}=4$	$SO(7)$	Yes	Yes
$d=4, \mathcal{Q}=4, \mathcal{N}=1$	$U(1)$	No	No
$d=4, \mathcal{Q}=8, \mathcal{N}=2$	$SO(2) \bigotimes SU(2)$	No	No
$d=4, \mathcal{Q}=16, \mathcal{N}=4$	$SO(6)$	Yes	Yes

One of the requirements that twisting procedure can be done is that one must start with sufficient SUSY in the continuum theory (2^d). Clearly maximal SUSY theories in $1 \leq d \leq 4$ satisfy this easily (great for holography!). Open problem to construct those which are not on the list. For an attempt of directly dealing with fine-tuning and studies of $\mathcal{N}=1$ SYM in four dimensions, see works by [Bergner, Münster, Montvay et al.] group.

Lattice $\mathcal{N} = 4$ SYM

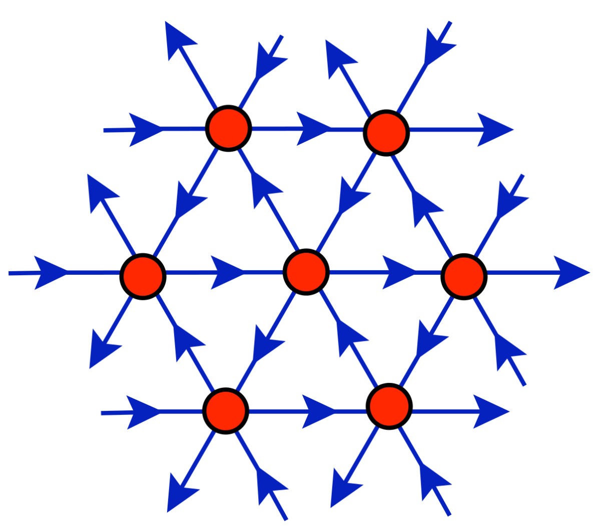
This talk will present results based on the geometric construction and idea of topologically twisting (maximal twist) a supersymmetric gauge theory. This generates the 0-form supercharges needed to preserve a subset of SUSY algebra. In some sense, this is just a way of rewriting original fields and is justified for flat Euclidean space. Supercharges are broken into p-forms and then put on the lattice sites, links, plaquettes respectively.

Basic idea: Take maximum subgroup $SO(4) \subset SO(6)$ of the R-symmetry group and construct $SO(4)_{\mathrm{tw.}} = \mathrm{diag}[\mathrm{SO}(4)_{\mathrm{E}} \times \mathrm{SO}(4)_{\mathrm{R}}]$. This gives a nilpotent (i.e., $Q^2 = 0$) supercharge which can then be preserved exactly on the lattice. If we look at the table, now it is clear why certain theories cannot be twisted in this sense!

Special lattice

- To reduce the fine-tuning to minimum possible and to identify the twisted fields in a consistent manner on the lattice, we cannot just work with the usual hypercubic lattice. We need what is called an A_4^* lattice. Four-dimensional version of the triangular lattice shown below.

- This lattice has S_5 point group symmetry and five links which is natural setting to lay out ten field components (complex gauge links) of the $\mathcal{N} = 4$ SYM theory. Note that basis vectors are not orthogonal.



Open access

✤ SUSY lattice community is small and large-scale lattice calculations are still in early days. We make our parallel, four-dimensional, arbitrary N code available on GitHub. Updated version release to follow on Computer Physics Communications (CPC) soon [Schaich, RGJ et al. as v2 of 1410.6971]

github.com/daschaich/susy

Dimensional reduction

Lattice $\mathscr{N}=4$ SYM is extremely hard and probably not possible to study at large N and λ in practice using modern techniques and algorithms. Also, there is a sign problem which we have observed for $\lambda \geq 5$. Additional complications because of being a super conformal field theory with no scales, moduli etc.

It is less complicated to consider lower-dimensional version(s) of this theory which are non-conformal and the computational costs are under control. Also, sign problem does not seem to play a role for range of couplings for interesting finite-temperature black hole Physics and matching to predictions for quantum `non-extremal' black holes.

Plan for today

We present our results for two models we have considered in recent works.

- ★ 2+1-dimensional SYM obtained by reducing the four-dimensional theory along one Euclidean direction.

- ★ 0+1-dimensional BMN matrix model which is a massive deformation of BFSS model. BFSS model can be thought of as $1d$ version of $\mathcal{N} = 4$ SYM.

3d $\mathcal{N} = 8$ SYM

The three-dimensional $(p = 2)$ SYM has a holographic description at large N and strong coupling. In this case, the black holes are known as black $D2$-branes. The weak coupling (high-temperature) thermodynamic behaviour is expected to be $\sim t^3$ by counting d.o.f. while the power-law behaviour of the energy density changes at strong coupling. Using the dual Type II supergravity metric, it is straightforward to compute the Hawking temperature (T), internal energy (E) and other thermodynamic quantities.

Thermodynamics

The internal energy and entropy associate with stack of N Dp-branes is given by (up to factors of spatial volume):

$$E = N^2 \frac{(9-p)\left((4\pi)^{\frac{2}{5-p}}\left(\frac{T\sqrt{\lambda 2^{7-2p}\pi^{\frac{1}{2}(9-3p)}\Gamma\left(\frac{7-p}{2}\right)}}{7-p}\right)^{\frac{2}{5-p}}\right)^{7-p}}{\lambda^2 2^{11-2p}\pi^{\frac{1}{2}(13-3p)}\Gamma\left(\frac{9-p}{2}\right)}$$

$$S = N^2 \frac{\sqrt{\lambda 2^{7-2p}\pi^{\frac{1}{2}(9-3p)}\Gamma\left(\frac{7-p}{2}\right)}\left((4\pi)^{\frac{2}{5-p}}\left(\frac{T\sqrt{\lambda 2^{7-2p}\pi^{\frac{1}{2}(9-3p)}\Gamma\left(\frac{7-p}{2}\right)}}{7-p}\right)^{\frac{2}{5-p}}\right)^{\frac{9-p}{2}}}{\lambda^2 2^{8-2p}\pi^{\frac{1}{2}(11-3p)}\Gamma\left(\frac{9-p}{2}\right)}$$

3d $\mathcal{N} = 8$ SYM

Translating these expressions to the action density (measured on the lattice and with $p = 2$) we get,

$$\frac{s_{\text{Bos}}}{N^2\lambda^3} \approx -0.831 t^{10/3} \qquad \frac{s_{\text{Bos}}}{N^2\lambda^3} \approx -2.598\ldots t^3$$

increasing $t = T/\lambda$

It is worth noting that for SYM on torus in $p+1$-dimensions we would have parametric dependence $s_{\text{Bos}} \propto t^{(14-2p)/(5-p)}$ for $t \ll 1$ from the gravity dual, and the $t \gg 1$ limit would go as $s_{\text{Bos}} \propto t^{p+1}$. In the $p = 3$ conformal case these powers coincide (for $\mathcal{N} = 4$ SYM).

Regime of lattice calculations

To have a valid supergravity (SUGRA) description we need:

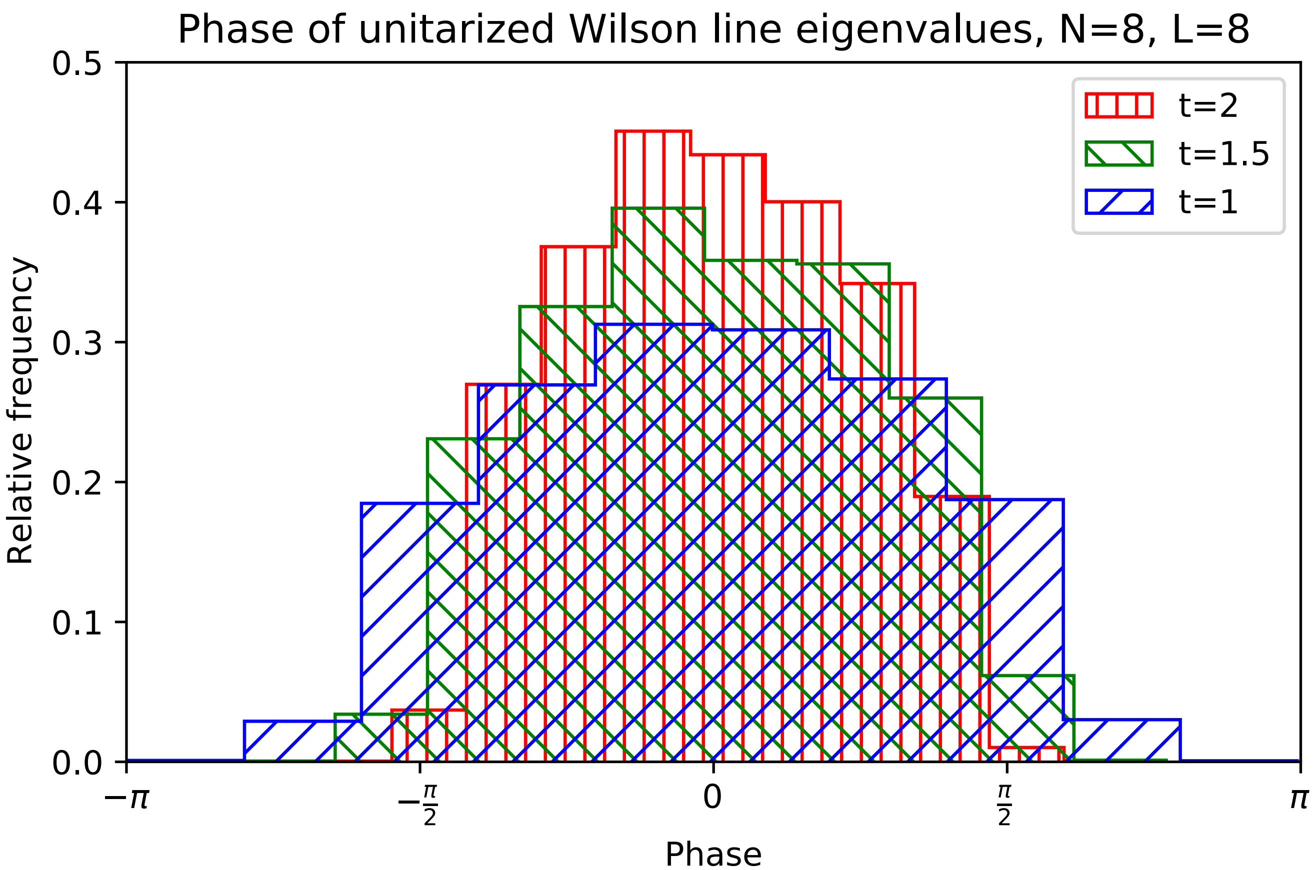
- Radius of curvature should be large in units of α'. This implies $1 \ll \lambda\beta^{3-p}$
- String coupling, g_s should be small

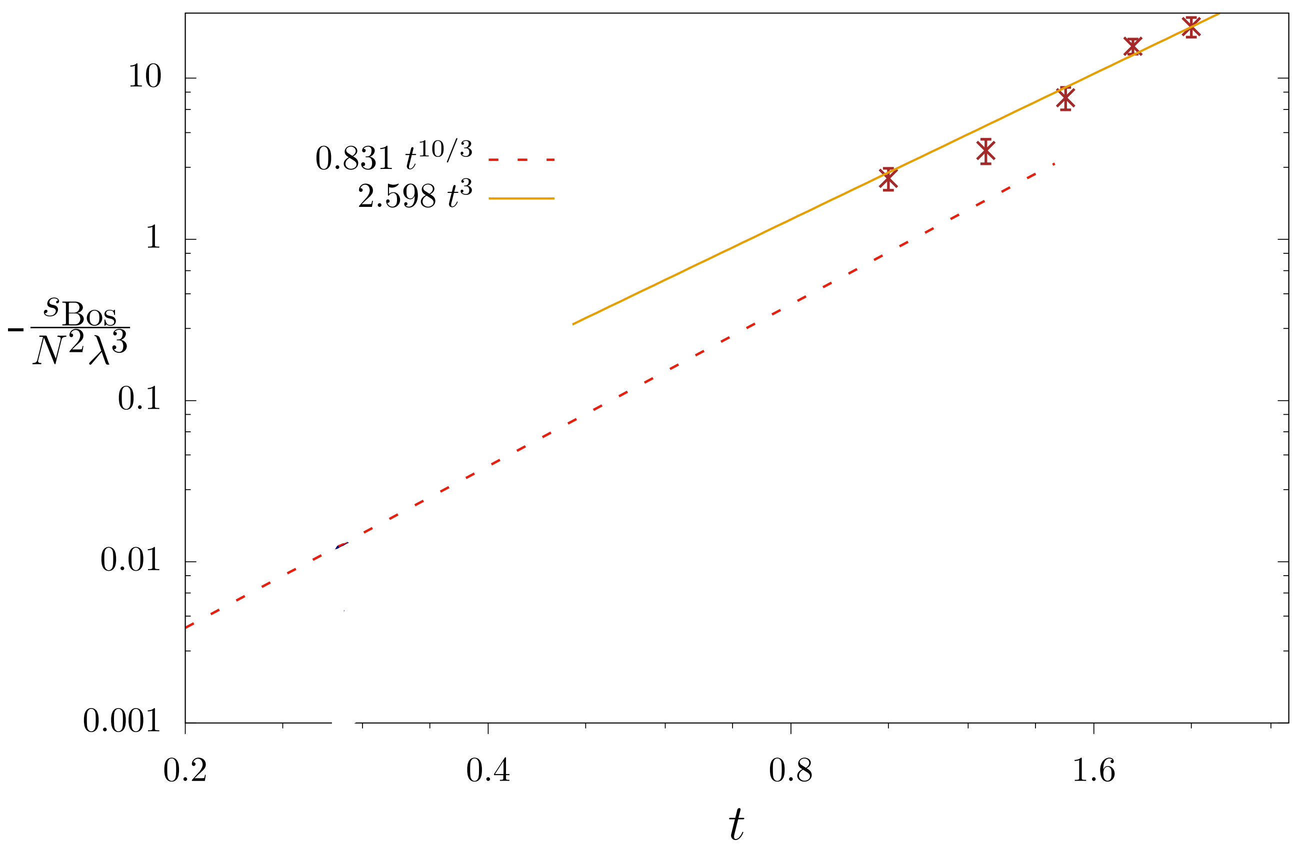
We can combine both conditions as:

$$1 \ll \lambda\beta^{3-p} \ll N^{\frac{10-2p}{7-p}}$$

Results - High temperature

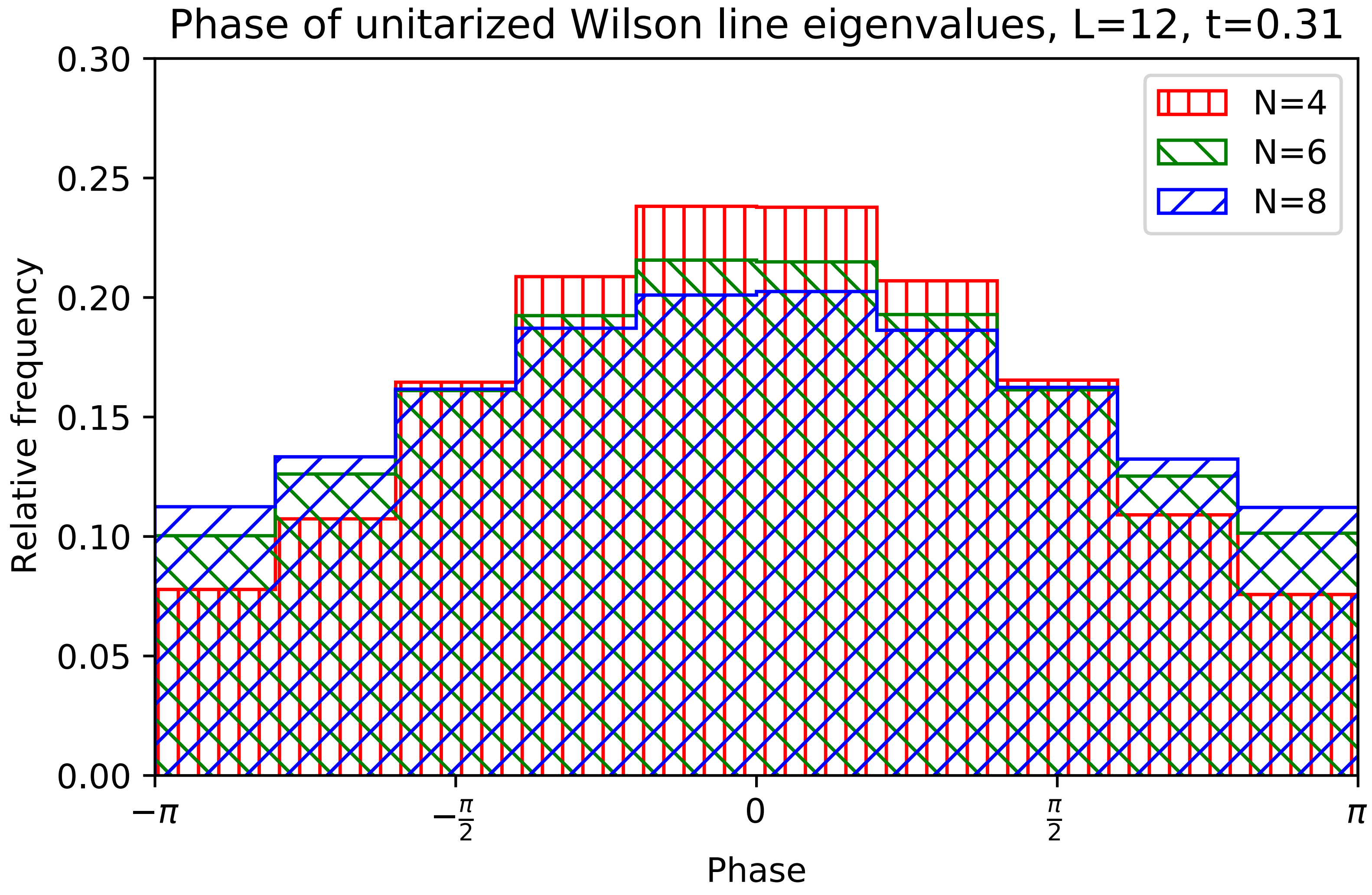
We start in the high-temperature phase and slowly reduce the temperature to approach the SUGRA regime (i.e., strong couplings). In our lattice computations, we explore maximum $N = 8$ and lattice volume of 16^3.





[Catterall, Giedt, RGJ, Schaich, Wiseman, 2010.00012]

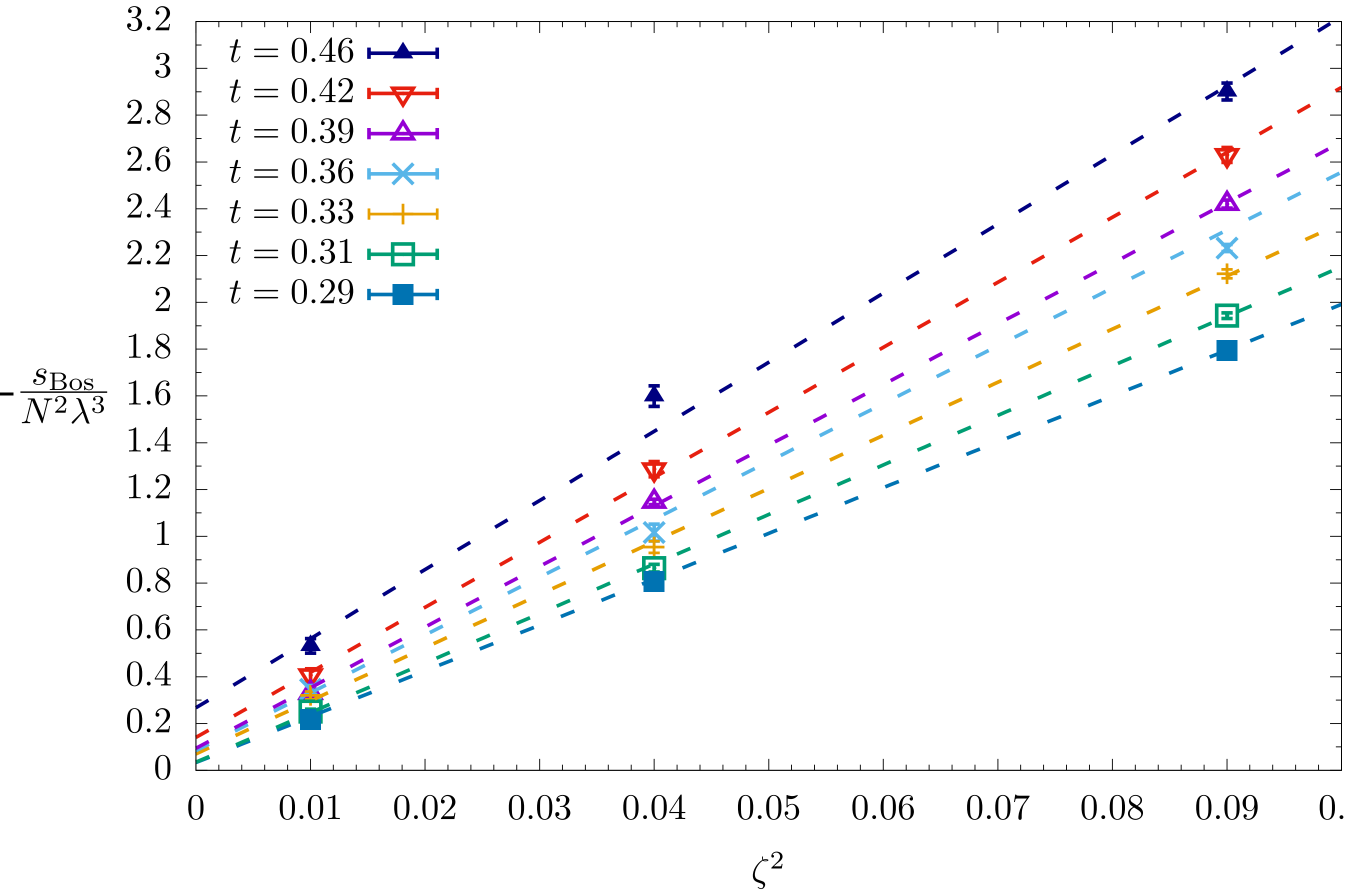
Low-temperature phase

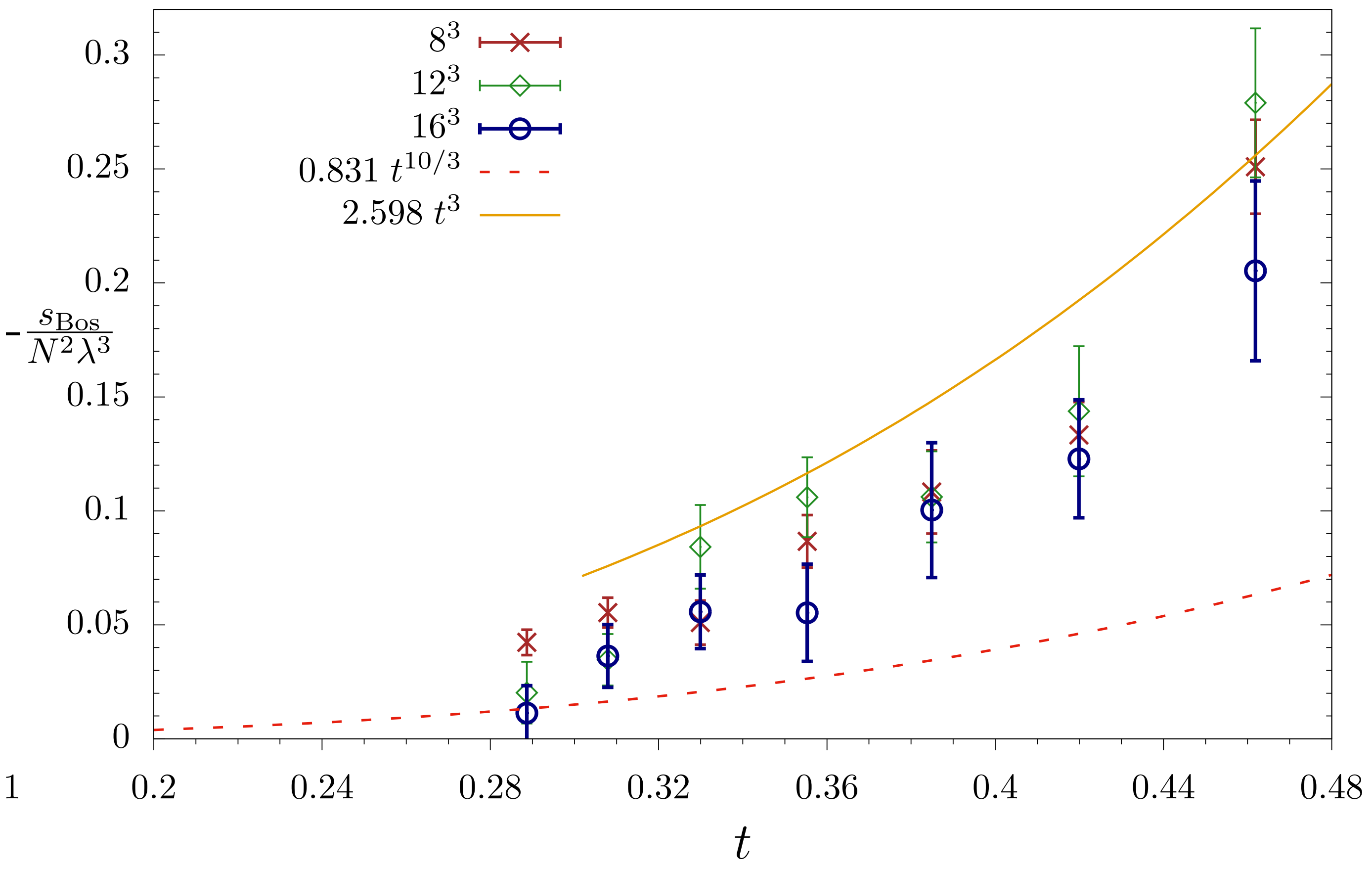


[Catterall, Giedt, RGJ, Schaich, Wiseman, 2010.00012]

Low-temperature (continued)

SUSY theories like the one considered here have well-known flat directions which render the numerics unstable. We have to introduce SUSY violating potential which depends on ζ^2. We extrapolate this to zero.





Future directions - I

We plan to explore the 3d $\mathcal{N} = 8$ SYM in more detail by exploring the supersymmetric Wilson loop. Computations in the gravity setting [Rey & Lee 98, Maldacena 98] have found out that $\ln W \sim \lambda^{1/3}$ in the large N limit. A lattice computation confirming this would be very interesting! Alternatively, understanding the rich-phase structure of this $3d$ SYM is another challenging problem.

Matrix models

Obtained by dimensional reduction of $\mathcal{N} = 1$ SYM from ten dimensions down to one. The models have $SU(N)$ gauge symmetry and $SO(9)$ internal symmetry group corresponding to nine scalars. Simplest holographic gauge theory with well-defined gravity dual.

$$S_{\text{BFSS}} = \frac{N}{4\lambda} \int dt \,\text{Tr}\Big[(D_t X^i)^2 - \frac{1}{2}\left[X^i, X^j\right]^2 + \Psi^T D_t \Psi + i\Psi^T \gamma^j [\Psi, X^j]\Big]$$

$$S_{\text{BMN}} = S_{\text{BFSS}} + S(\mu)$$

μ-terms break the $SO(9) \to SO(6) \otimes SO(3)$. BFSS has a single deconfined phase but BMN model admits a *deconfinement* phase transition!

BFSS := Banks-Fischler-Shenker-Susskind [1996]

BMN := Berenstein-Maldacena-Nastase [2002]

BMN matrix model

$$S_{BMN} = S_{BFSS} - \frac{N}{4\lambda}\int d\tau \ \mathrm{Tr}\left(\frac{\mu^2}{3^2}\left(X^i\right)^2 + \frac{\mu^2}{6^2}\left(X^M\right)^2 + \frac{2\mu}{3}\epsilon_{IJK}X^iX^jX^k + \frac{\mu}{4}\overline{\Psi}^\alpha\left(\gamma^{123}\right)_{\alpha\beta}\Psi^\beta \right)$$

The flat directions of the BFSS model are lifted by giving masses to $SO(3)$ and $SO(6)$ scalars. In addition, there is a cubic scalar term which is also known as 'Myers term' plus a fermion term. Dual gravity solution applicable when $g = \lambda/\mu^3 \gg 1$ with $\mu \ll 1$, $N \to \infty$.

Exact results for $g \to 0$

In this limit, the theory can be studied perturbatively. Note that since $g = \dfrac{\lambda}{\mu^3}$, this is the large μ limit. In this case, the model becomes a supersymmetric gauged Gaussian model. It was well-studied and the critical temperature was determined to be:

$$\left.\frac{T}{\mu}\right|_c = \frac{1}{12\ln 3}\left[1 + O(\lambda) + O(\lambda^2)\right]$$

which increases with λ but is bounded by some gravity result as we will see soon. For $\lambda = 0$, we have $(T/\mu)_c \approx 0.076$.

[See for details: O'Connor et al. 1805.05314 and Schaich, RGJ, Joseph in 2003.01298]

Exact results for $g \to \infty$

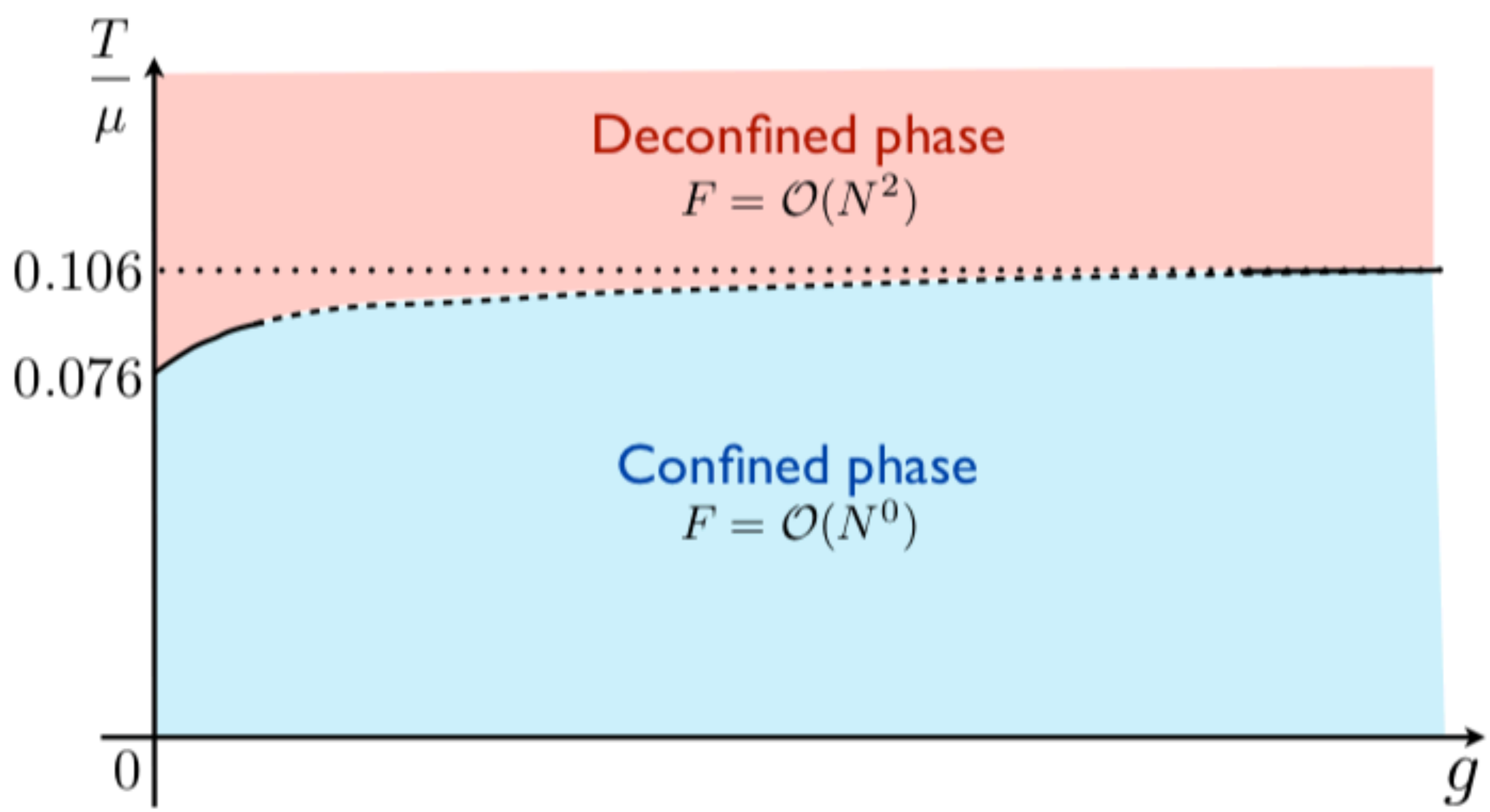
In this limit, the theory can be studied in the dual gravity setting. Though the zero-temperature Type IIA solutions for BMN model is known via work of Lin, Lunin, and Maldacena, it is hard to do this for finite temperatures. However, one challenging computation done in [1411.5541 by Costa, Penedones, Santos et al.] computed the critical temperature in that limit.

$$\left.\frac{T}{\mu}\right|_c \approx 0.106$$

However, there is not much known for finite g and that is what we want to understand using the lattice.

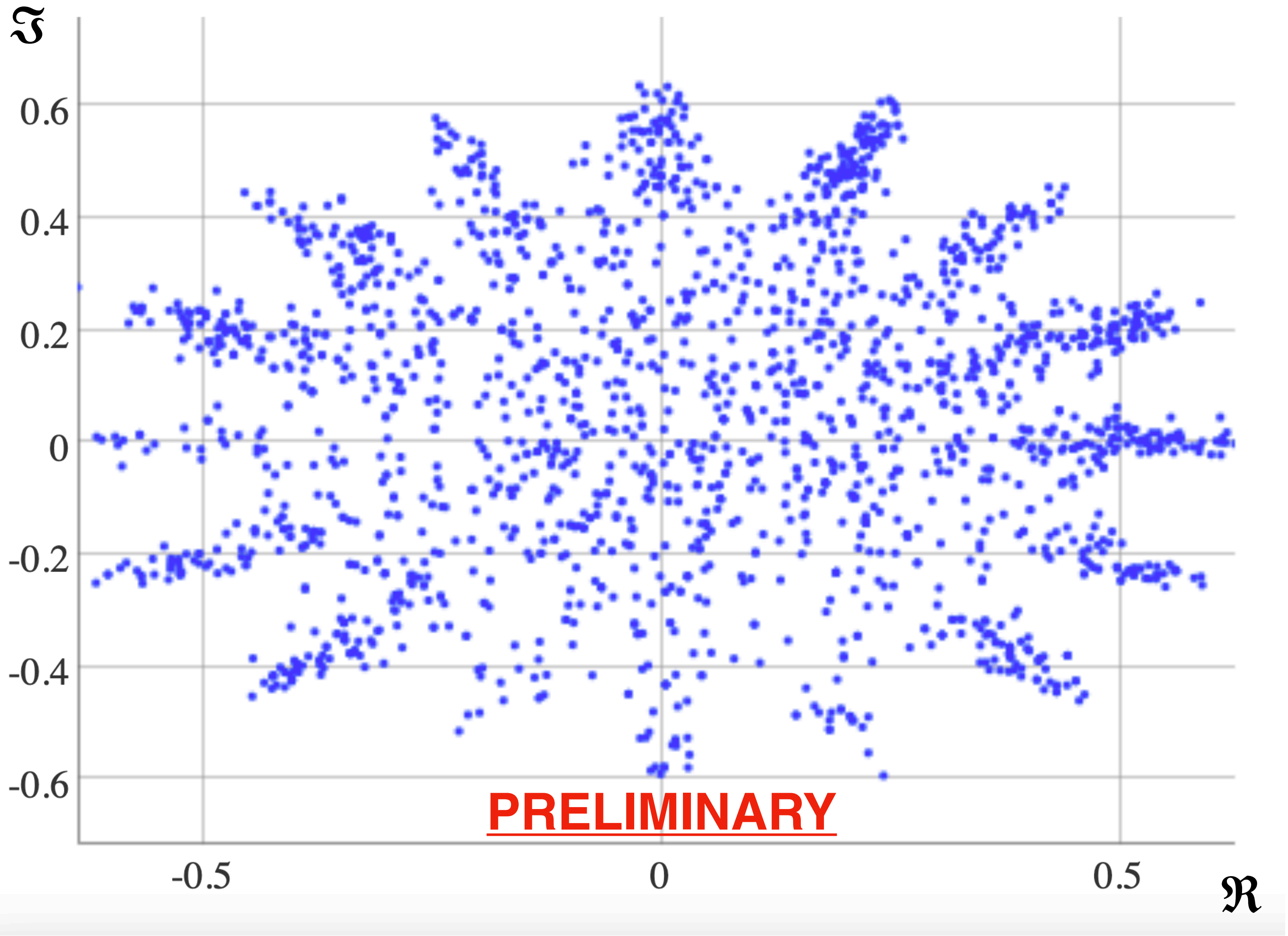
Conjectured phase diagram

For this model, the phase structure should look like one given below. The solid lines are known results from perturbation theory and gravity computations. The dashed lines are educated guess. There might also be non-smooth behaviour unlike what is shown. Seems like a simpler version of the more complicated $'3/4'$ behaviour in finite-temperature $\mathcal{N}=4$ SYM.

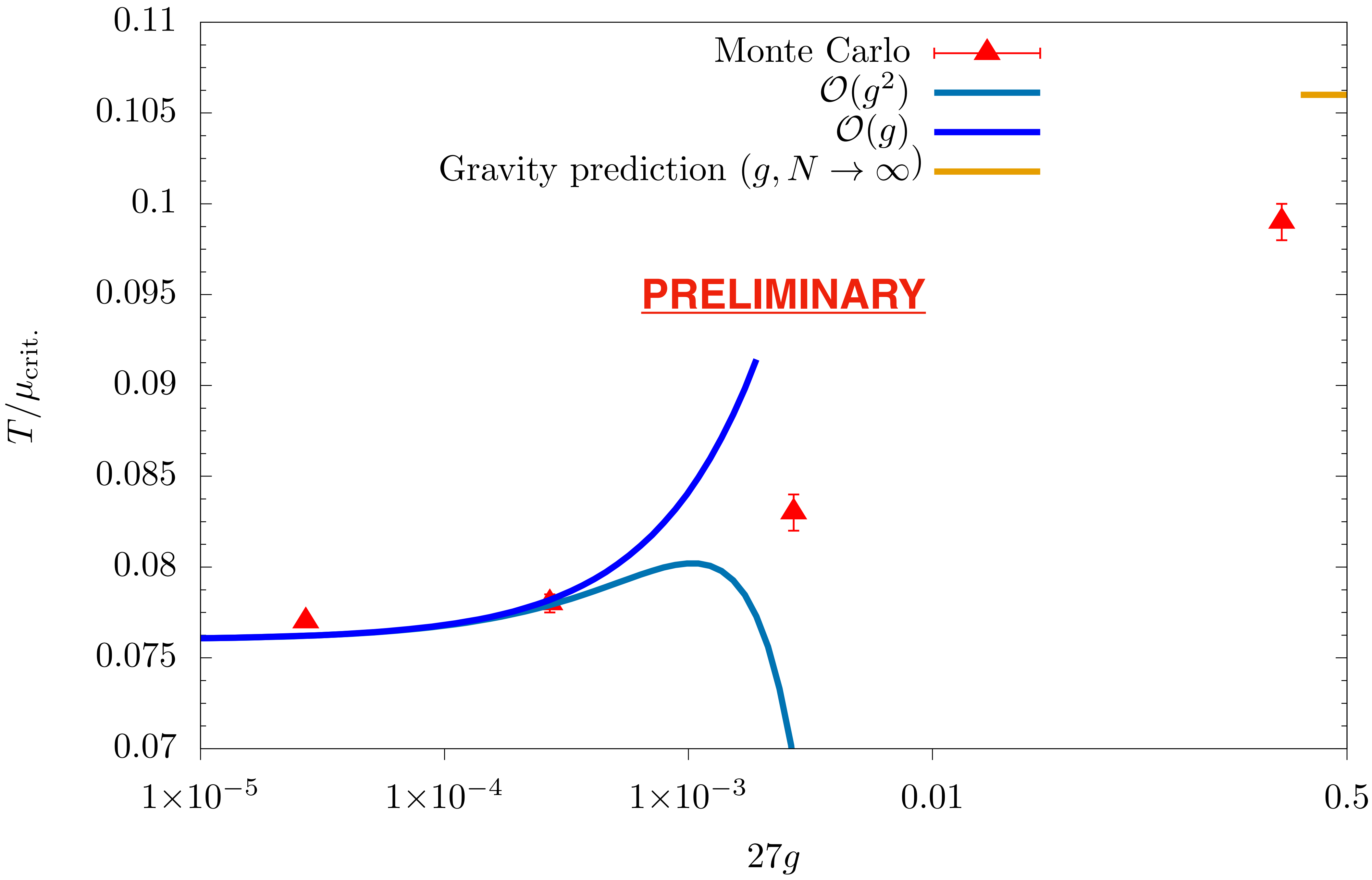


1411.5541

Polyakov loop scatter plot: SU(16)



Phase diagram



Challenges

We have only been able to explore till $N = 16$ and with that we can probe a maximum $g \sim 0.2$ which is $\mu \sim 1.71$ (in units where $\lambda = 1$). This is still only the intermediate regime between perturbative and supergravity. To access holography in the sense approached by 1411.5541 we need to ensure $\mu \ll 1$ which needs larger N. The computational complexity is $\approx \mathcal{O}(N^{7/2})$ which is hard! In the future, parallelizing the software over matrix degrees of freedom seems essential.

Future directions - II

BMN matrix model is a massive deformation of the BFSS matrix model. In the past decade there has been much progress [Catterall, Wiseman, Hanada, O'Connor and others] in understanding the thermodynamics of this model and good agreement has been seen with dual supergravity calculations.

However, this program has not been much successful for the BMN model since the dual gravity computations at finite-temperatures are hard to come by and there is no exact expression to compare to. But, this should not be an obstruction to making some claims based on lattice computations. For ex: we know that for BFSS in the large N limit: $E/N^2 \sim 7.41\ T^{14/5}$. For BMN, this is modified to $E/N^2 \sim 7.41\ T^{14/5} f(\mu, T)$. Lattice computations should be able to compute $f(\mu, T)$ which would be valuable in constructing/understanding finite-temperature black hole solutions.

Future directions - II

	Full BMN	Bosonic sector
Gauged BMN	(0.076, 0.106)	(0.089, ??)
Ungauged BMN	??	??

Work in progress!

(a, b) denotes critical temperature in the $g \to 0$ and $g \to \infty$ limits.

Thank you