Tensor networks *and* spin models

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Different RG methods

Different renormalization group (RG) methods have been introduced over the past 5-6 decades:

- Kadanoff's spin blocking RG [1966] & Wilson's RG [1975]
- Density Matrix Renormalization Group (DMRG) [White, 1992]
  (DMRG is a refined extension to above approach and is well-suited to all 1d systems not only restricted to impurity problems such as Kondo problem.)
- Tensor Renormalization Group [Levin and Nave, 2007] + HOTRG [Xie et al., 2012]
  (More efficient than DMRG but breaks down at criticality as former.)
- Tensor Network Renormalization (TNR) [Vidal and Evenbly, 2015]
  (TNR is an extension of TRG which qualitatively improves TRG behaviour for systems at criticality and can be used to generate MERA tensor networks.)

Rev. Mod. Phys. 47, 773 (1975)

The fourth aspect of renormalization group theory is the construction of nondiagrammatic renormalization group transformations, which are then solved numerically, usually using a digital computer. This is the most exciting aspect of the renormalization group, the part of the theory that makes it possible to solve problems which are unreachable by Feynman diagrams. The Kondo problem has been solved by a nondiagrammatic computer method. The renormalization group solution of the Kondo problem is explained in detail in this paper: see Sec. VII–IX. The two dimensional Ising model has been solved approximately by several nondiagrammatic ("block spin") renormalization group methods, by Niemeyer and Van Leeuwen (1973, 1974, 1975) and others. An example is detailed in Sec. VI. The Ising calculation is only a practice calculation, since the exact solution is known. Recently, Kadanoff (1975) and Kadanoff and Houghton (1975) have developed very powerful block spin methods which have been applied to the three dimensional Ising model, with considerable success.

Advantages of tensors?

- Provides an arena for studying lower-dimensional critical and gapped systems faster and more efficiently than any other numerical method available! [2d Ising model in 20 seconds!]

- Formulating in terms of tensors can enable us to study systems where the usual numerical methods (such as Monte Carlo fail due to sign problem!). In addition, the partition function is directly accessible in the thermodynamic limit unlike MC methods where we can only get expectation values.

- Description in terms of Matrix Product States (MPS) etc. is useful for real-time dynamics such as scattering of particles etc.

- Known to play an important role in understanding the AdS/CFT (i.e., bulk physics from entangled quantum state at the boundary).

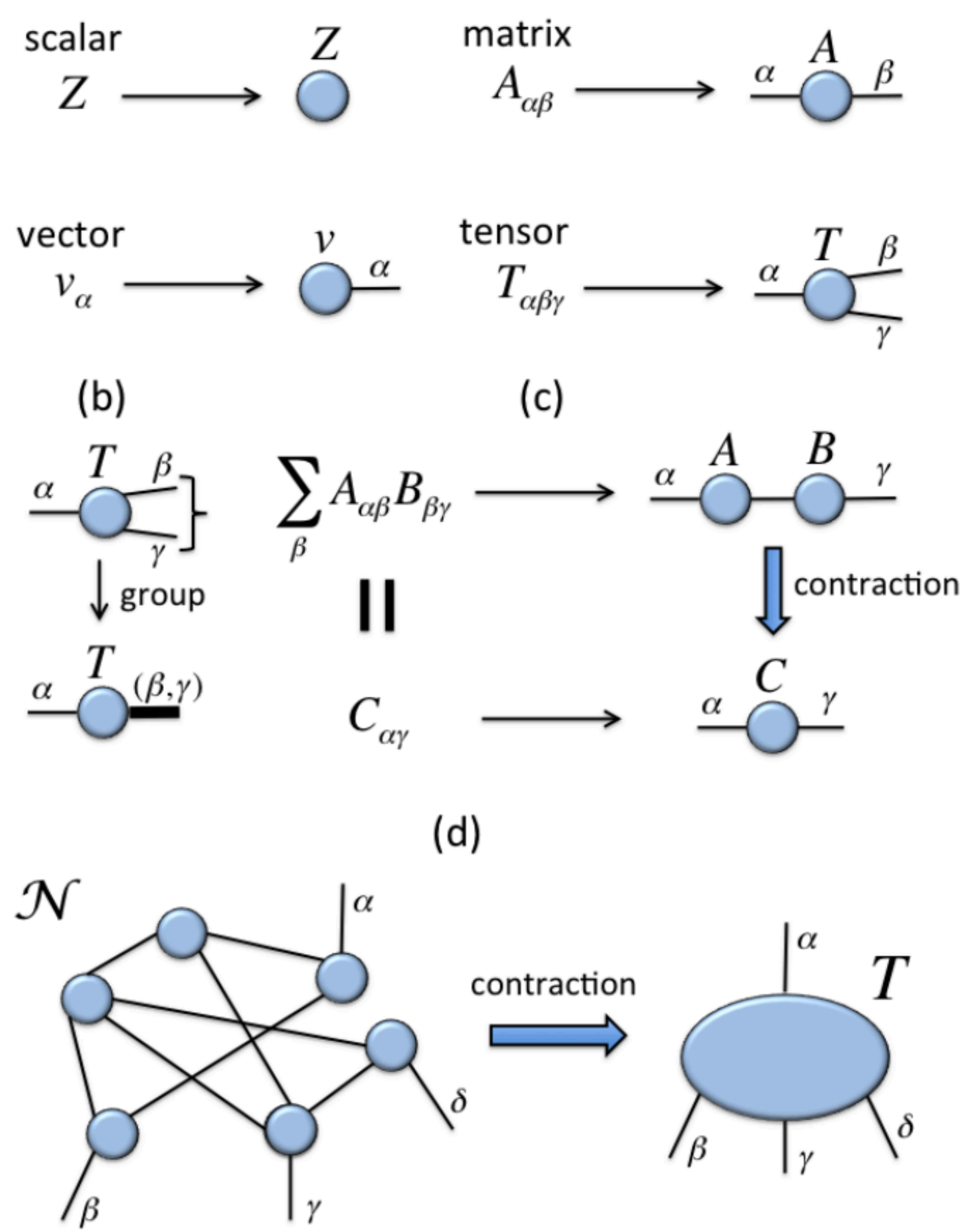
Different approaches

- Tensor networks approach belong to two categories: Lagrangian and Hamiltonian approaches. For ex:

  $|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} C_{i_1 \cdots i_N} | i_1 i_2 \cdots i_N \rangle$ as approximation to the ground state wave function of complicated many-body quantum system with local Hamiltonian. Ex: Matrix Product States (MPS) representation. Reduction to $O(N)$ rather than $O(d^N)$ coefficients.

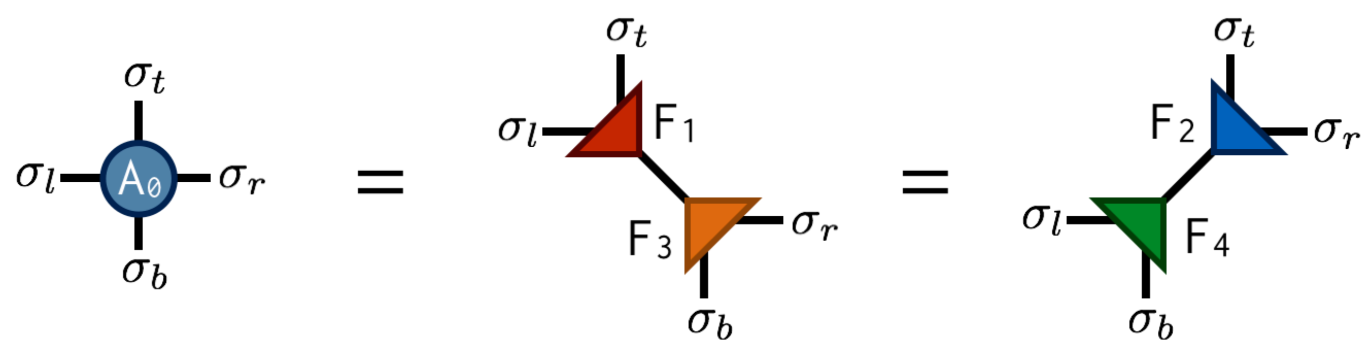
  $Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})}$ to approximate in the Lagrangian formulation (like we consider later).

Notation


scalar
matrix
vector
tensor
(b)
(c)
group
contraction
(d)
contraction

Standard TRG

Assuming that the system is represented by a building block i.e. rank-4 tensor given by A_0. The first move is to do singular value decomposition (SVD) of this as shown below.



This is shown in steps as

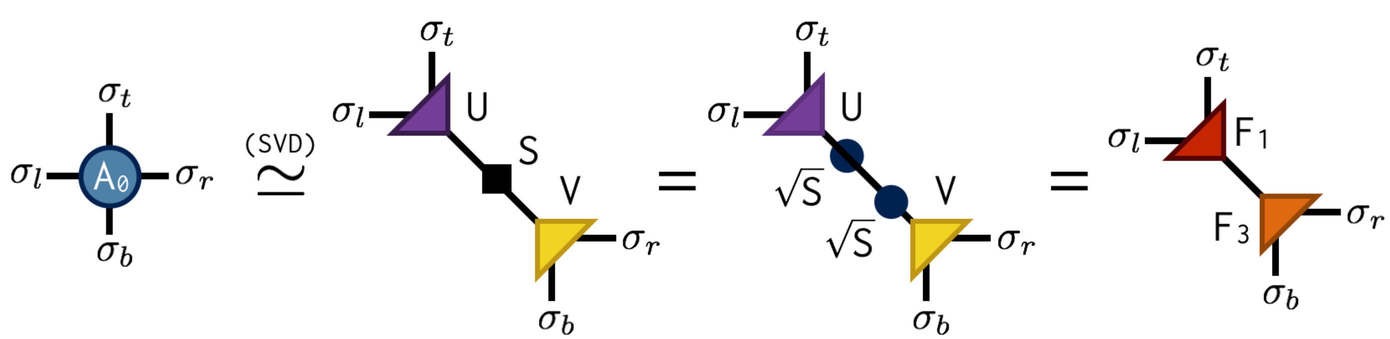


Figure: http://tensornetwork.org/trg/

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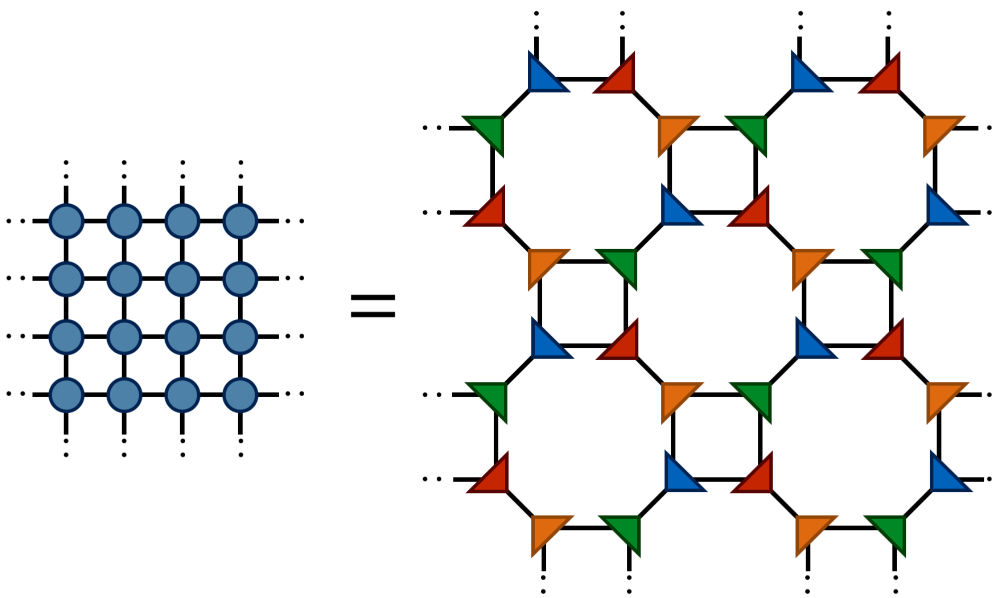
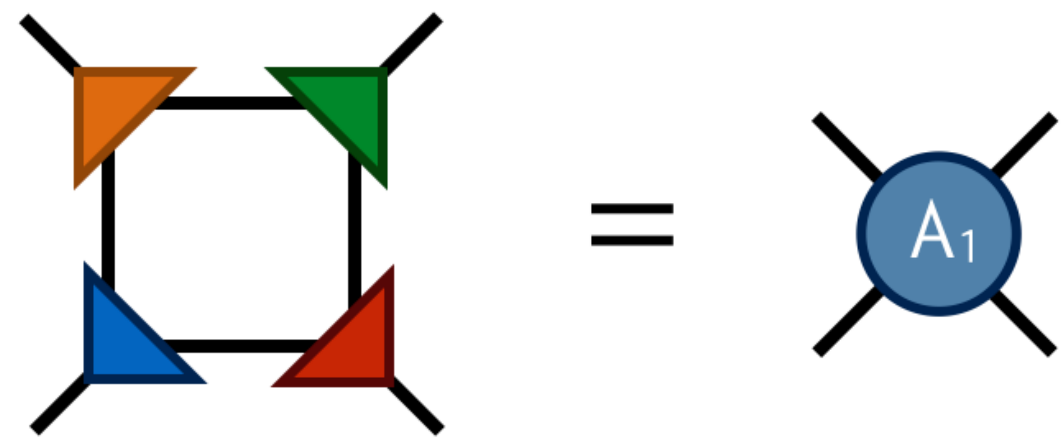
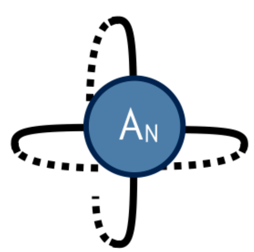






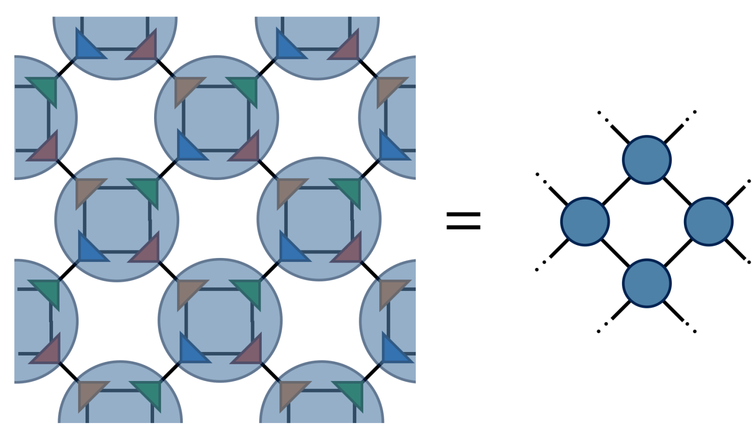


Figure: http://tensornetwork.org/trg/

Higher-order TRG

A refined real space coarse graining method similar in spirit to TRG but employs higher-order SVD (HOSVD) to reduce the errors due to truncation. First introduced in arXiv:1201.1144 and is successfully applied to statistical systems in $d = 2, 3$ and recently also in four dimensions on an advanced supercomputer. Performs better than naive TRG for critical systems. Less complex than the TNR methods.

Coarse-graining renormalization by higher-order singular value decomposition

Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, T. Xiang

We propose a novel coarse graining tensor renormalization group method based on the higher-order singular value decomposition. This method provides an accurate but low computational cost technique for studying both classical and quantum lattice models in two- or three-dimensions. We have demonstrated this method using the Ising model on the square and cubic lattices. By keeping up to 16 bond basis states, we obtain by far the most accurate numerical renormalization group results for the 3D Ising model. We have also applied the method to study the ground state as well as finite temperature properties for the two-dimensional quantum transverse Ising model and obtain the results which are consistent with published data.

2d Ising Model [Square lattice]

Exactly solvable system with solution due to Onsager, where the logarithm of the partition function is given by:

$$f(\beta) = -\frac{1}{\beta}\left(\ln(2) + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\ln\left[2\cosh^2(2\beta) - \sinh(2\beta)\cos(\phi_1) - \sinh(2\beta)\cos(\phi_2)\right]d\phi_1 d\phi_2\right)$$

And has continuous phase transition at:

$$T_c = \frac{2}{\ln(1+\sqrt{2})} = 2.26918531421 \implies \beta_c \approx 0.440687$$

We apply tensor methods to this system and see how we do.

2d Ising Model [Square lattice]

The fundamental tensor can be written down as:

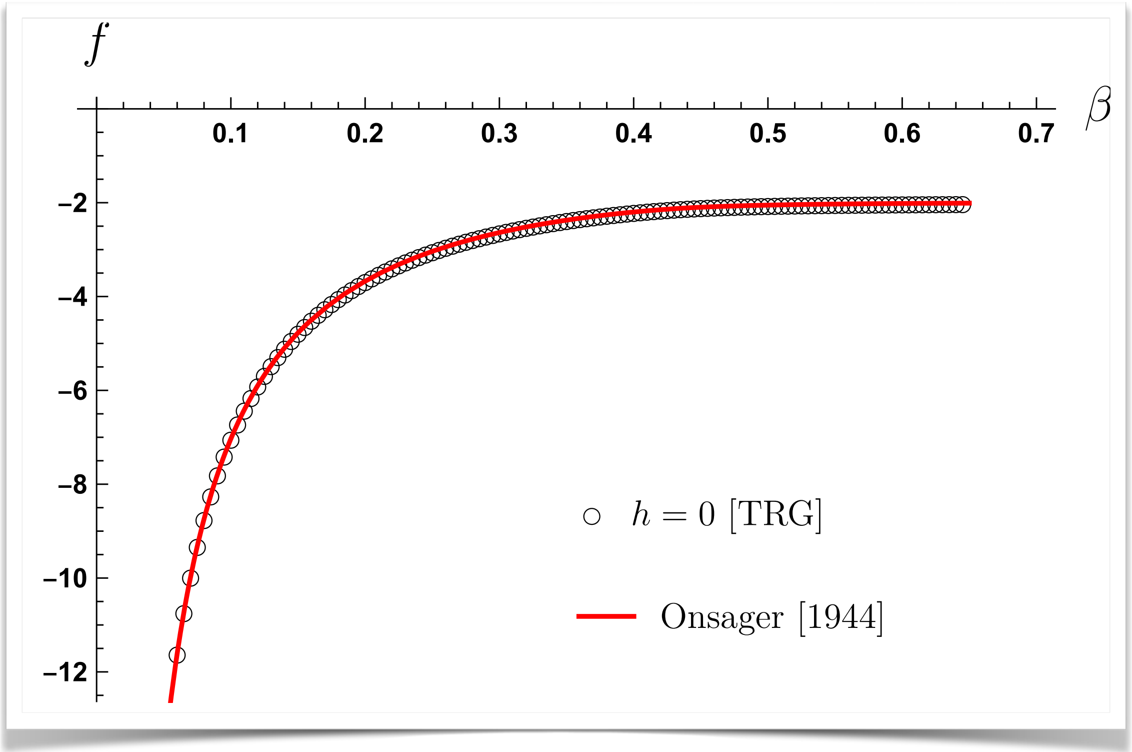
$$T_{abcd} = W_{ia} W_{ib} W_{ic} W_{id}$$

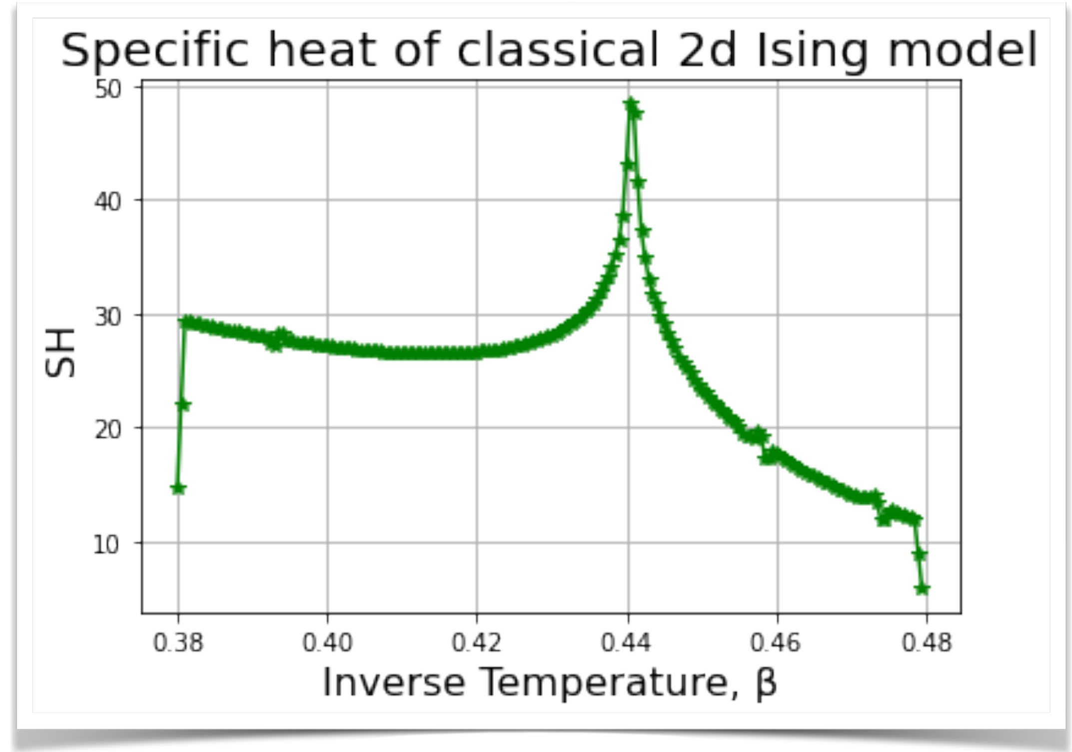
with W given by a 2×2 matrix [decomposing Boltzmann weights]:

$$W_{ia} = \begin{bmatrix} \sqrt{\cosh(\beta)} & \sqrt{\sinh(\beta)} \\ \sqrt{\cosh(\beta)} & -\sqrt{\sinh(\beta)} \end{bmatrix}$$

W can also be easily modified to include h to study model in magnetic field.

Solution - 30 seconds on laptop





Ising model with $h \neq 0$

Ising model on random graph with magnetic field was solved in 1986 by Kazakov & Boulatov by mapping to a Hermitian matrix model, but there is no solution for any regular lattice for generic h! But, doing similar exercise with tensors is straightforward and takes few minutes. It is even simpler in some sense since there is no phase transition when h is non-zero!

On a square lattice if we define $z = \exp[-2\beta h]$, then Onsager case is $z = 1$, while Yang-Lee [1952] conjectured and Merlini [1974] gave expression for the free energy for $h = i\pi/2\beta$, i.e., $z = -1$.

$$f\left(\beta, \frac{i\pi}{2\beta}\right) = -i\frac{\pi}{2} - \frac{1}{\beta}\left(\ln 2 + \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln\left[\sinh^2(2\beta)\left(1 + \sinh^2(2\beta) + \frac{\cos(\phi_1+\phi_2) - \cos(\phi_1-\phi_2)}{2}\right)\right] d\phi_1 d\phi_2\right).$$

On the solution of the two-dimensional ising model with an imaginary magnetic field βH=h=iπ/2

D. Merlini

Lettere al Nuovo Cimento (1971-1985) **9**, 100–104 (1974) | Cite this article

Ising model in a magnetic field - Square lattice

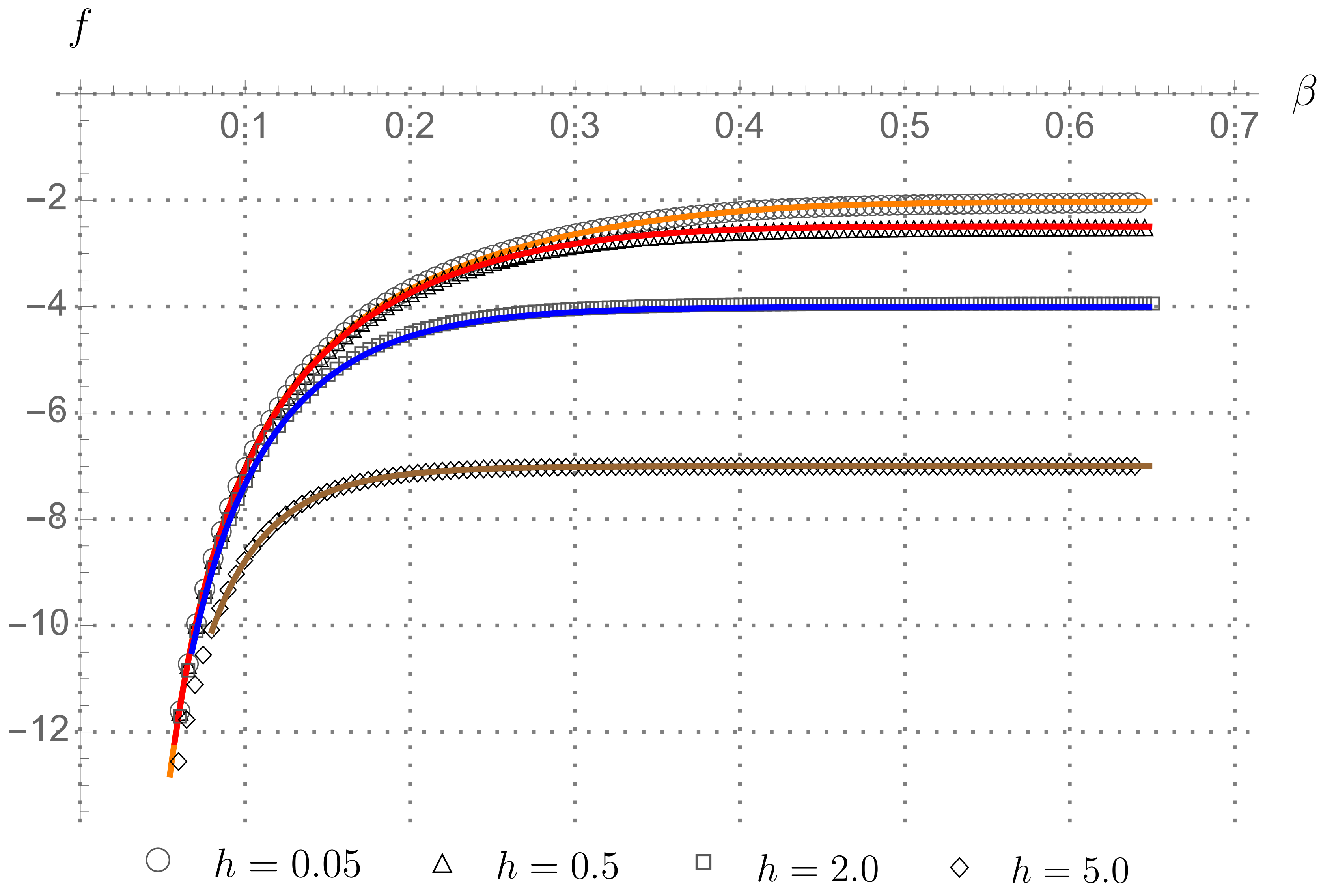
$$f(\beta, h) = ?$$

The tensor description is given by:

$$W_{ia} = \begin{bmatrix} e^{\Gamma}\sqrt{\cosh(\beta)} & e^{\Gamma}\sqrt{\sinh(\beta)} \\ e^{-\Gamma}\sqrt{\cosh(\beta)} & -e^{-\Gamma}\sqrt{\sinh(\beta)} \end{bmatrix}$$

$$T_{abcd} = W_{ia}W_{ib}W_{ic}W_{id}$$

Ising model in a magnetic field - Result



2d O(2) model

Simplest spin model with continuous symmetry in two dimensions. The nearest neighbour Hamiltonian is given by:

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) - h \sum_i \cos\theta_i$$

In order to construct the tensor representation, we decompose the Boltzmann weight (for say $h = 0$) using Jacobi-Anger expansion as:

2d O(2) model

The partition function can then be written as:

$$Z = \int \prod_i d\theta_i \prod_{\nu_{ij},\mu_i} I_{\nu_{ij}}(\beta) I_{\mu_i}(\beta h) e^{i\nu_{ij}(\theta_i - \theta_j) + i\mu_i\theta_i}$$

By integrating over $d\theta_i$, we obtain the initial tensor for XY model

$$T_{ijkl} = \sqrt{I_i(\beta)I_j(\beta)I_k(\beta)I_l(\beta)}\, I_{i+k-j-l}(\beta h)$$

2d O(2) model

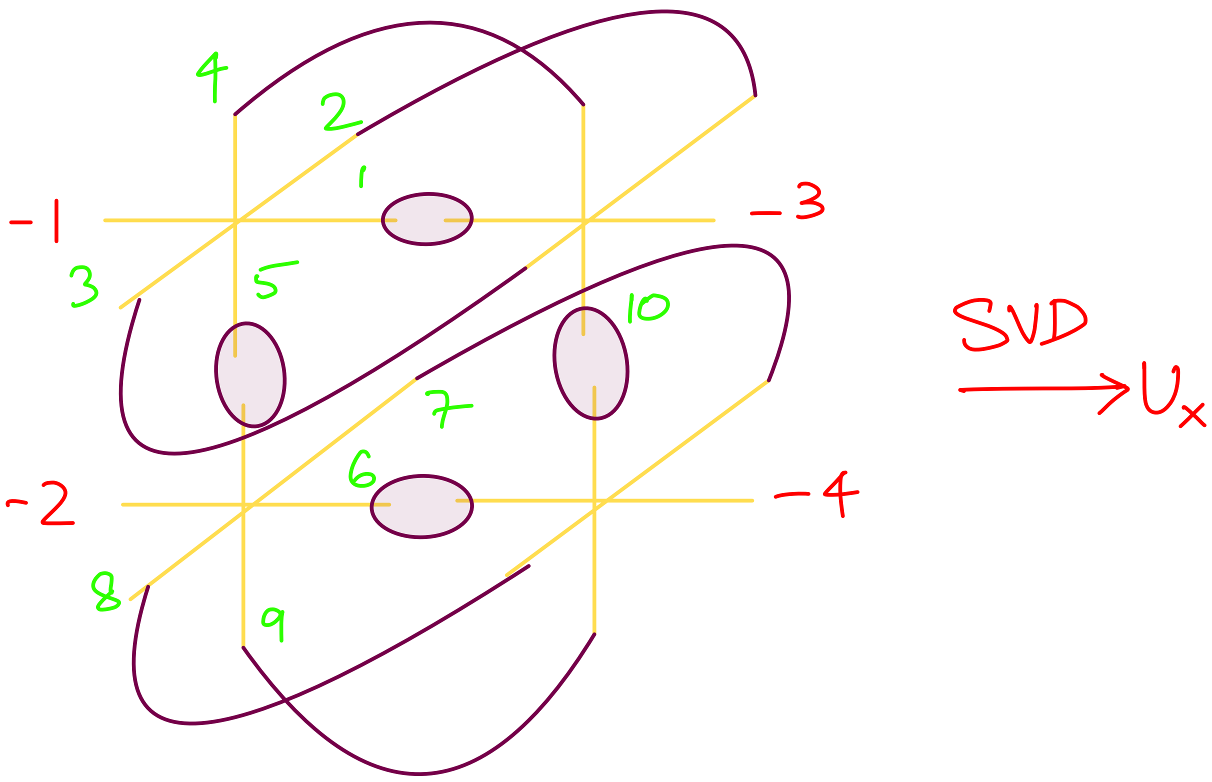
METHOD	YEAR	SYSTEM SIZE	T_{critical}
Monte Carlo [21]	1992	$2^9 \times 2^9$	0.89400(500)
HTE [22]	1993	–	0.89440(250)
Monte Carlo [23]	1995	$2^8 \times 2^8$	0.89213(10)
Monte Carlo [24]	2005	$2^{11} \times 2^{11}$	0.89294(8)
HTE [25]	2011	–	0.89286(8)
Monte Carlo [15]	2012	$2^{16} \times 2^{16}$	0.89289(5)
Monte Carlo [26]	2013	$2^9 \times 2^9$	0.89350(10)
Higher-order TRG [7]	2013	$2^{40} \times 2^{40}$	0.89210(190)
Uniform MPS [8]	2019	–	0.89300(10)
Higher-order TRG [This work]	2020	$2^{50} \times 2^{50}$	0.89290(5)

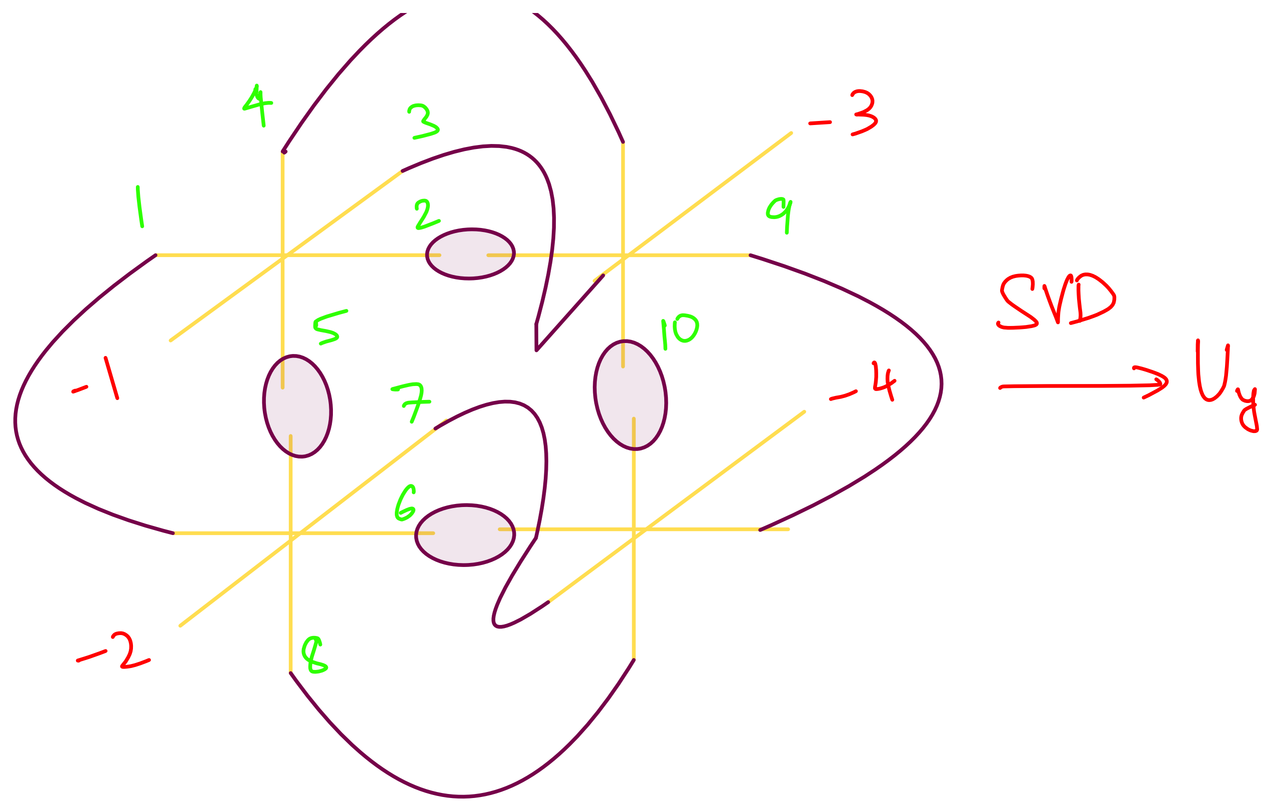
TN

Moving to 3d models - why is it tough?

There are several reasons: 1) The errors which we accumulate renders any reliable RG procedure inefficient. 2) The growth of entanglement is much more and if this is not removed by some well-defined procedure method, one will never see the fixed-point behaviour, 3) The computational time scales as: $\mathcal{O}(D^{4d-1})$ [next slide]

Bottleneck - Singular value decomposition (SVD)

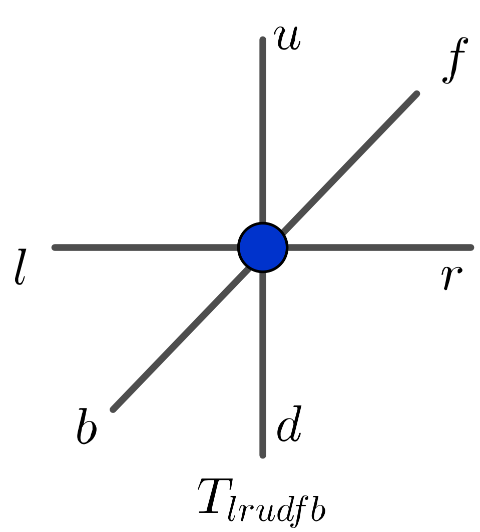


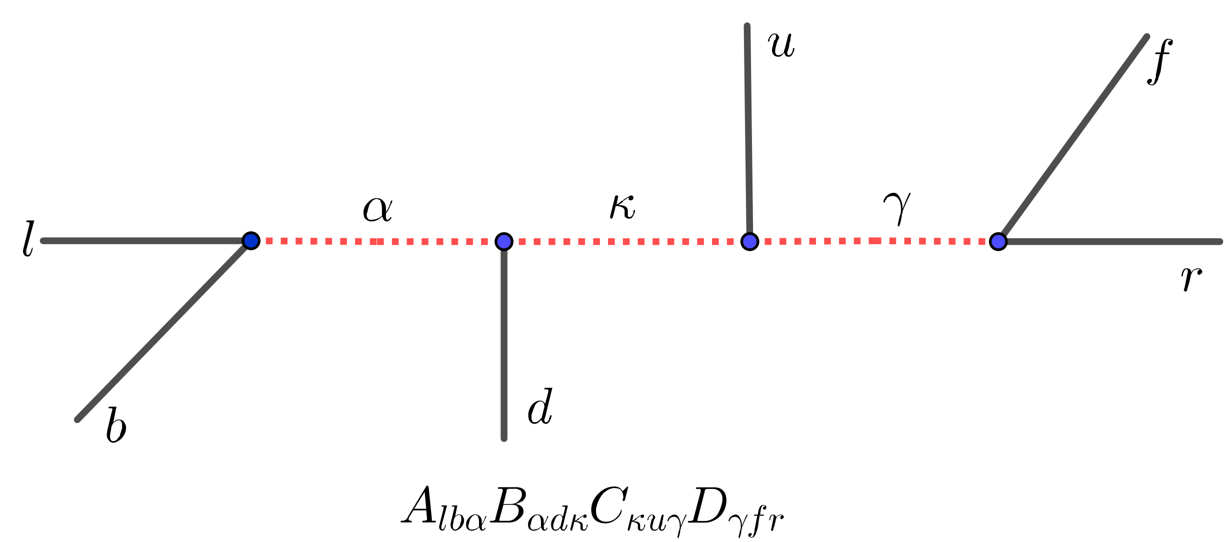


Triad decomposition

However, we can try and see how close we can get to say other methods [Monte Carlo, Bootstrap, etc.]

For that we have to first deal with reducing $O(D^{4d-1})$, and this was done by Kadoh et al. in 2019, this is referred to as `triad' method. The computational cost is reduced to $O(D^{d+3})$ if RSVD is used or else $O(D^{d+4})$.





3d classical spin models with tensors

- Ising model well-studied but still critical exponents not computed!
- $O(2)$ model First study: arXiv: 2105.08066 [RGJ et al]
- $q > 3$ Potts model *First study: In preparation, RGJ (2022),*

3d O(2) model

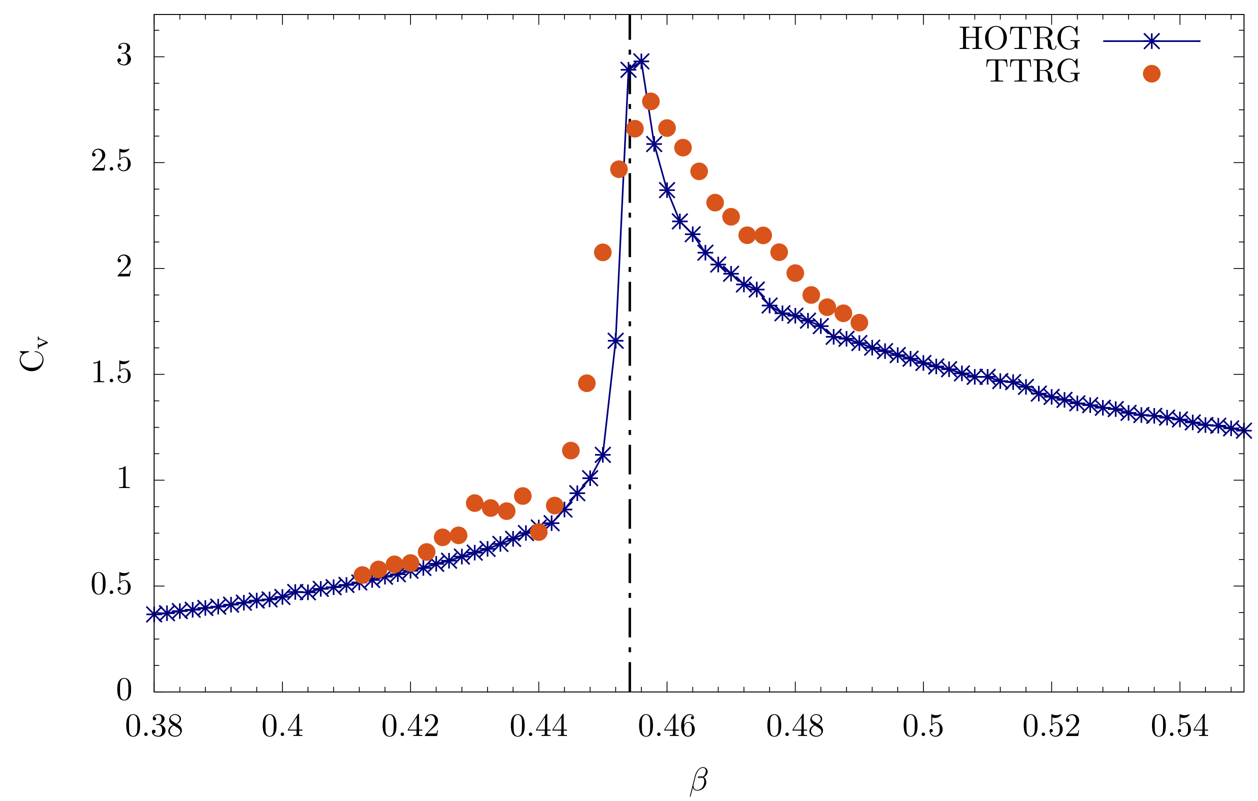
The partition function can then be written as:

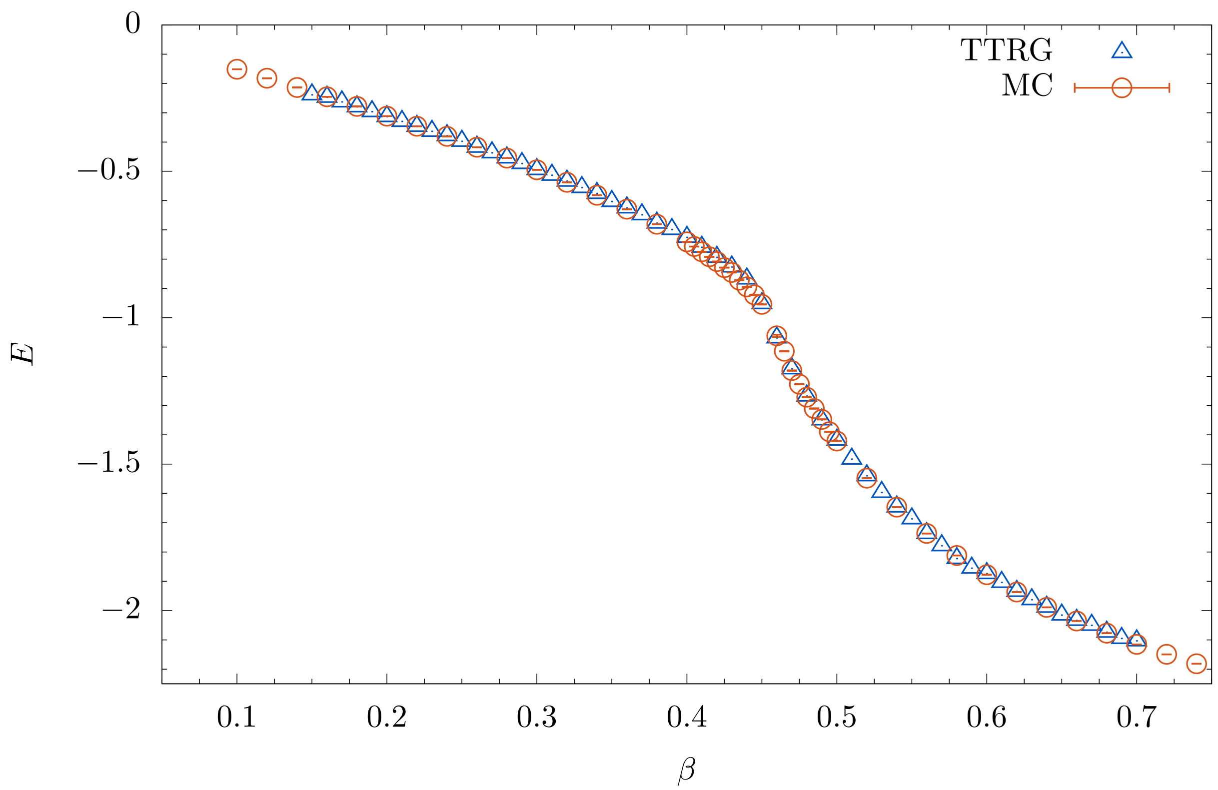
$$Z = \exp[-S], \quad S = -\beta \sum_{j} \sum_{\nu=0}^{2} \cos(\theta_j - \theta_{j+\hat{\nu}}) - \beta h \sum_{j=1}^{V} \cos \theta_j$$

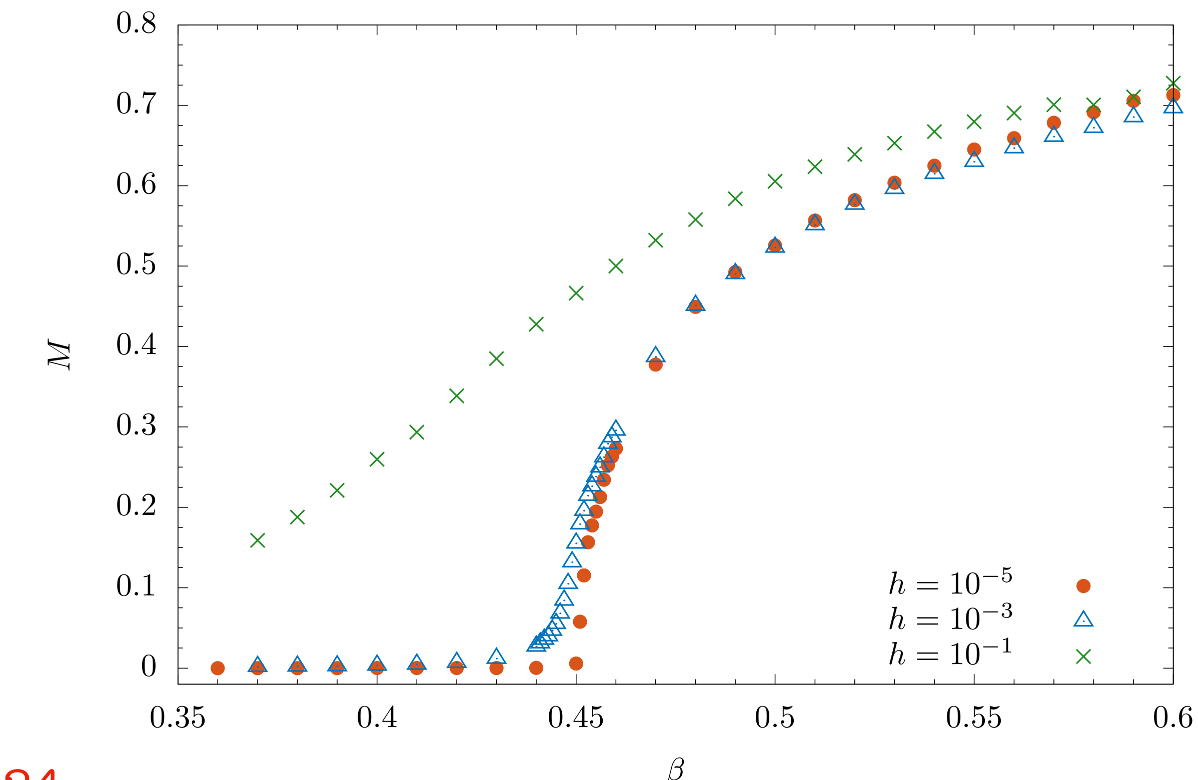
The initial local tensor obtained (as before) by decomposing Boltzmann weight is given by:

$$T_{ijklmn} = \sqrt{I_i(\beta) I_j(\beta) I_k(\beta) I_l(\beta) I_m(\beta) I_n(\beta)}\, I_{i+k+m-j-l-n}(\beta h)$$

3d O(2) model — Results







3d O(2) model — far from CB

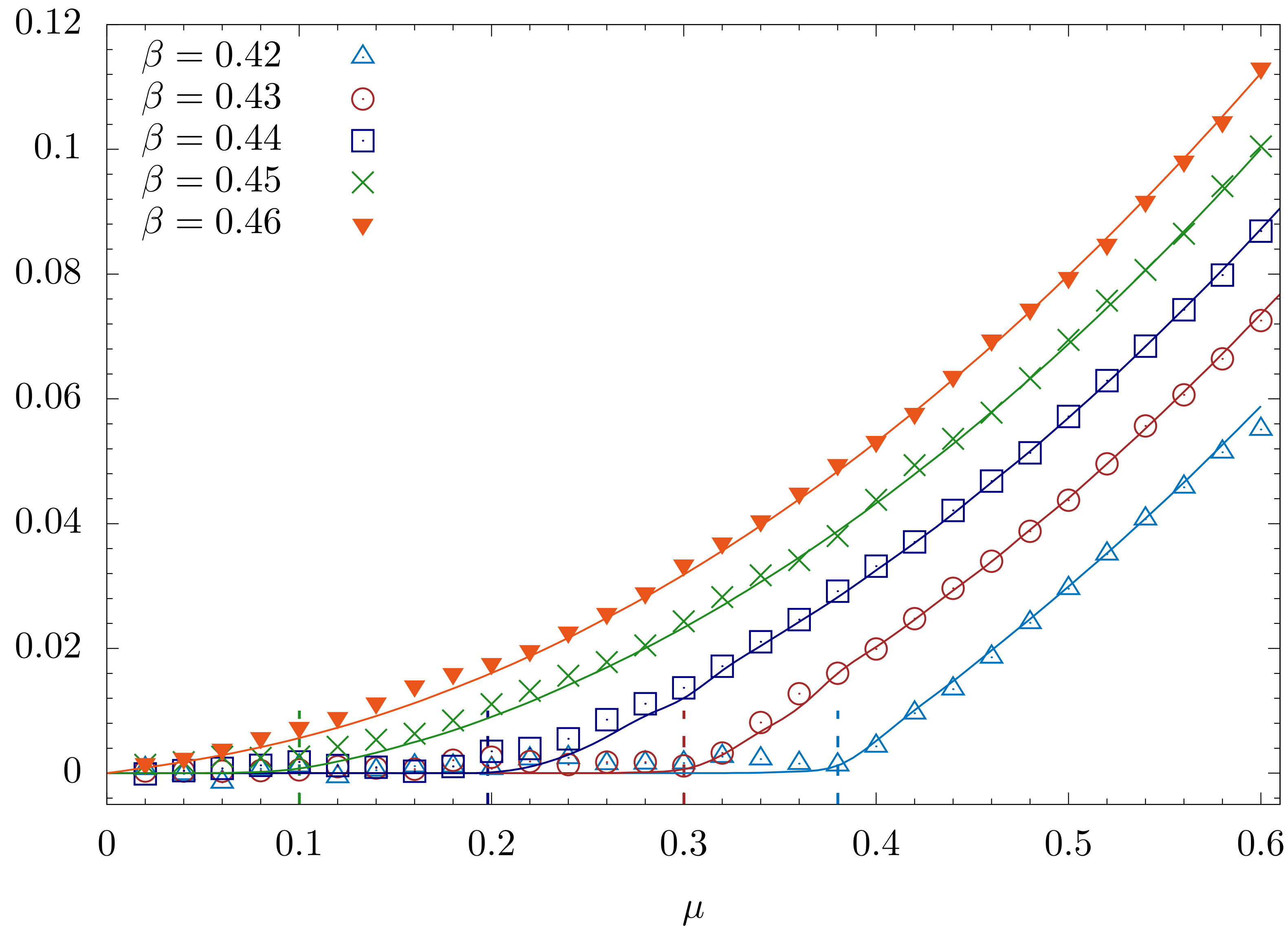
The tensor methods we used was good enough to study thermodynamical observables but were not accurate enough to compute coefficients. So, matching to Monte Carlo and CB for scaling dimensions of operators at critical point seems several years away and would some way of optimizing the triad algorithm we have explained.

Chemical potential [Sign problem!]

$$S = -\beta \sum_{j} \sum_{\nu=0}^{2} \cos(\theta_j - \theta_{j+\hat{\nu}} - i\mu\delta_{\nu,0}) - \beta h \sum_{j=1}^{V} \cos\theta_j$$

This model cannot be studied by usual Monte Carlo methods since the action is complex. Tensor network show promise for studying these systems and also with topological θ-term. A very long-term goal is to study finite-density QCD!

3d O(2) model - Silver Blaze (Zero temp.)



$$\frac{\partial \ln Z}{\partial \mu}$$

q-state Potts model

Generalization of Ising model ($q = 2$) model. These models have been studied for over 70 years. Some exact solutions are available in $2d$, but only numerical results are known in $3d$. They have phase transitions for all $q > 2$ in two and more dimensions. In $2d$, the transition is continuous for $q \leq 4$ and first-order for $q > 5$ with $q = 5$ being weakly first-order. In $3d$, this $q_\star = 2.45(5)$ has been numerically computed. The strength of the first-order transition increases with q (more latent heat!). In fact, these models have also been extensively studied for non-integer "q".

q-state 3d Potts model

RGJ, in preparation

The Hamiltonian is given by:

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j),$$

and the partition function:

$$Z = \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} \exp\left[\beta \delta(\sigma_i, \sigma_j)\right]$$

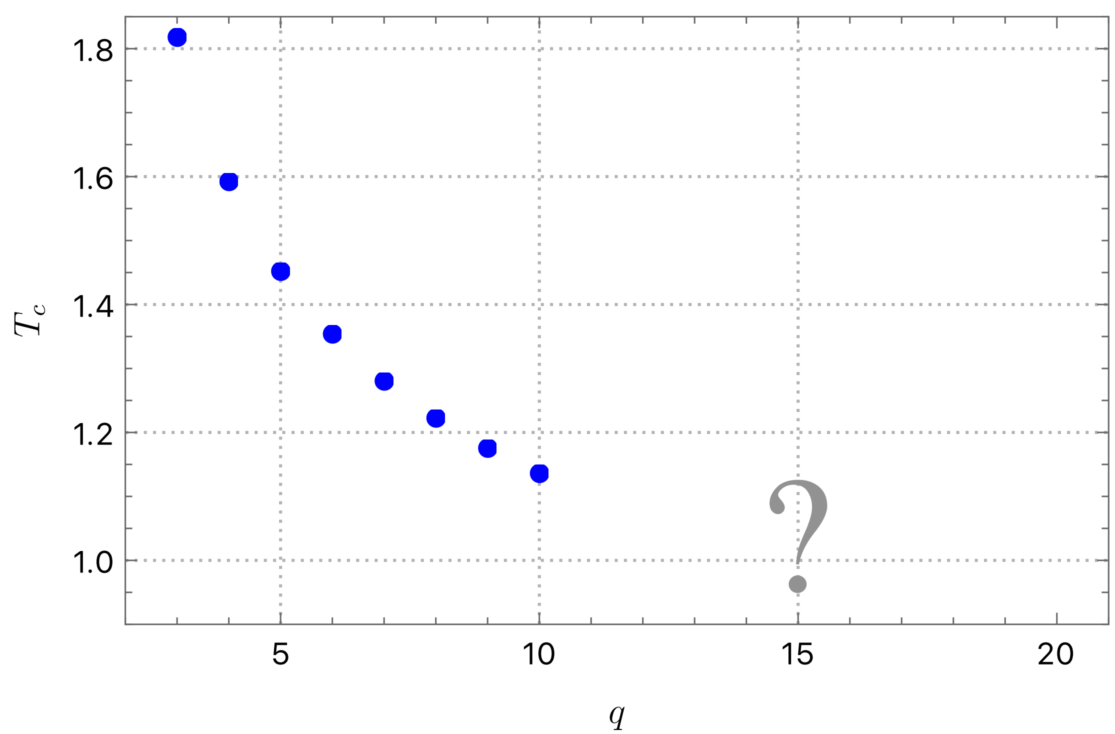
We construct the tensor to coarse-grain as follows:

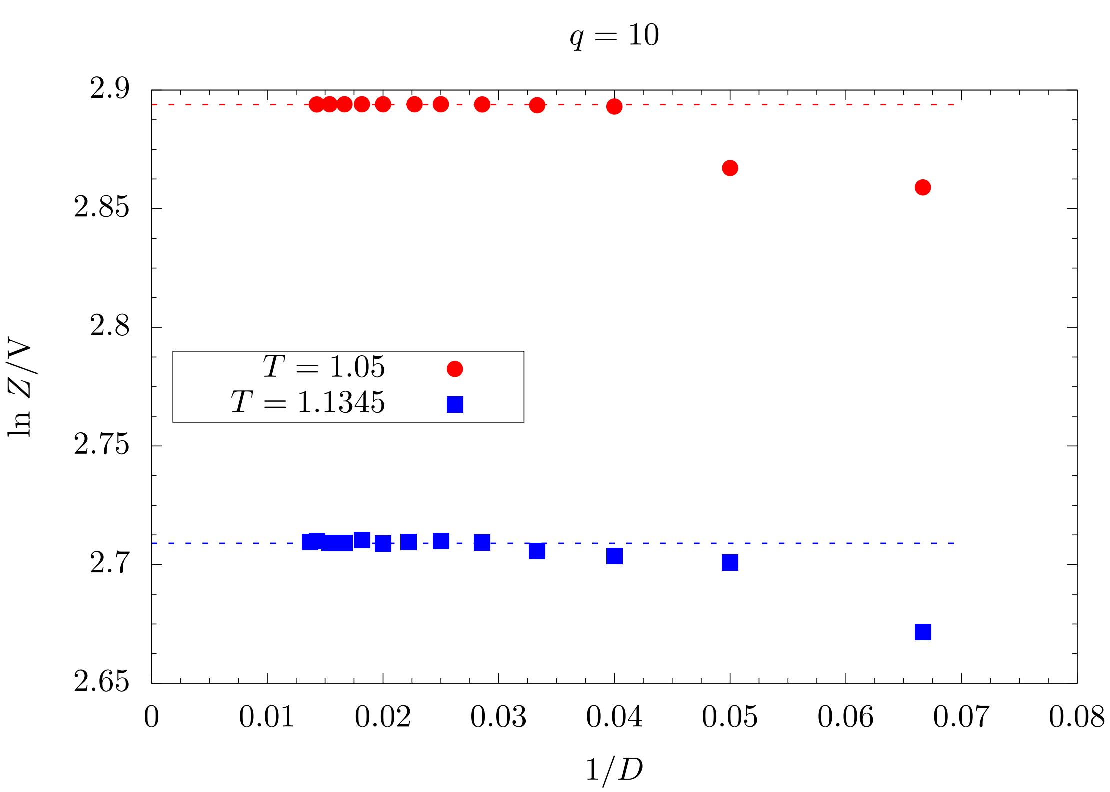
$$\mathbb{W}_{ij} = e^{\beta}, \text{if i = j}$$
$$= 1, \text{ if i} \neq \text{j}$$

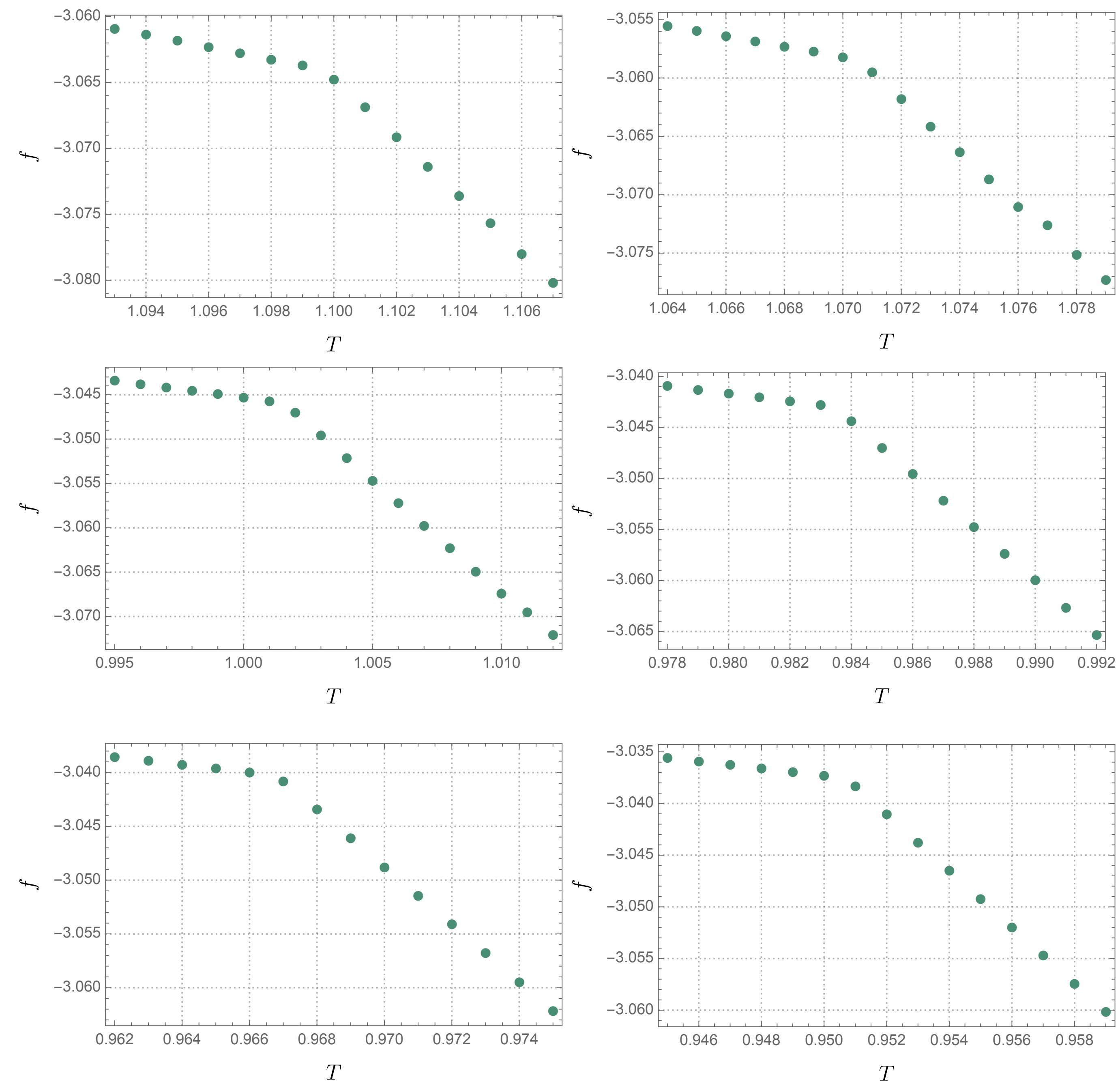
$$\mathbb{W} = \underbrace{QQ^T}_{\text{Cholesky}}$$

$$A_{xya} = Q_{ax}Q_{ay} \qquad B_{azb} = \mathbb{I}_{ab}Q_{az} \qquad C_{bz'c} = \mathbb{I}_{bc}Q_{bz'} \qquad D_{cy'x'} = Q_{cy'}Q_{cx'}$$

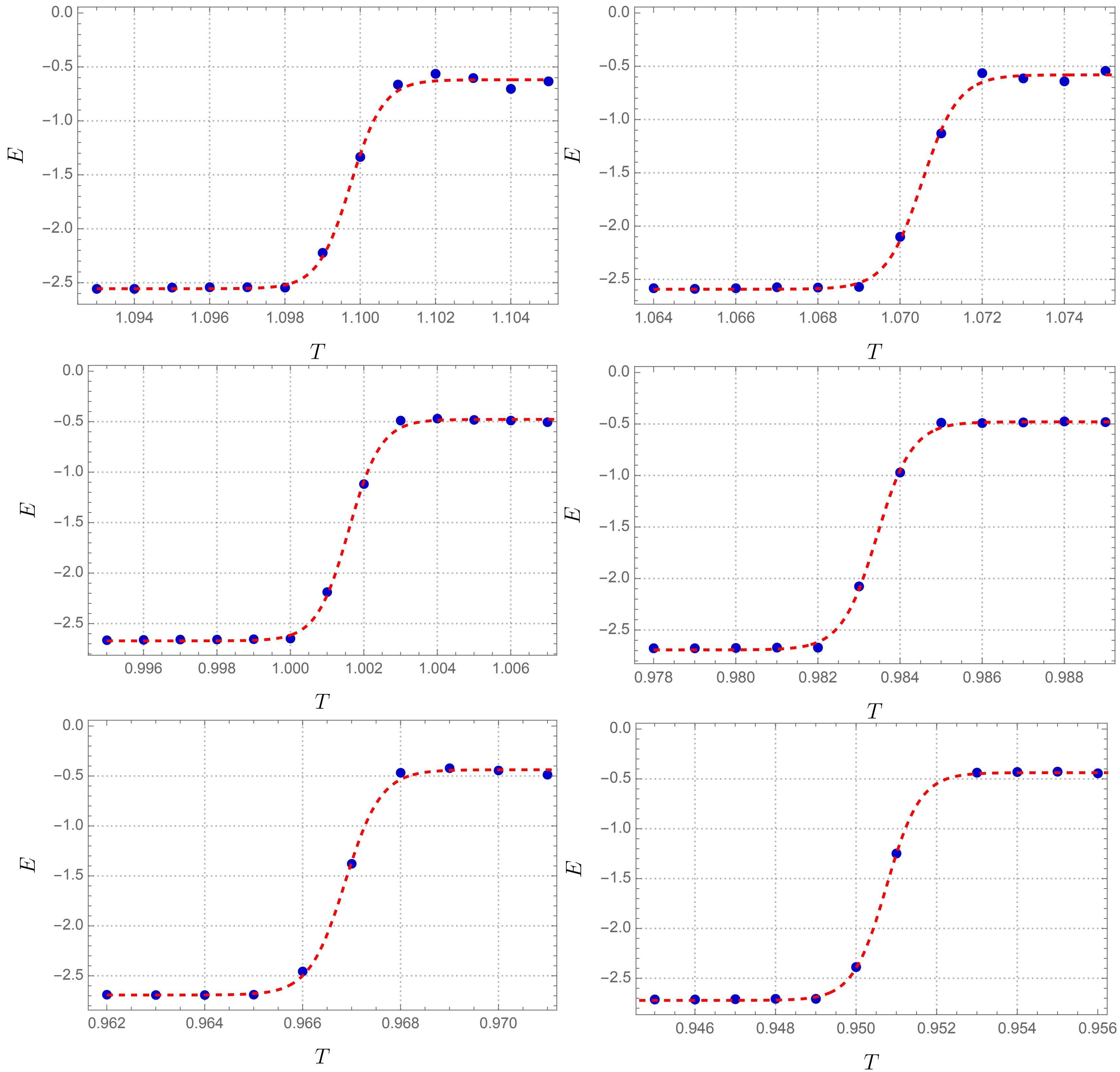
q-state Potts model - *Plan and results*







q-state Potts model



q-state 3d Potts model

q	$T_{c,\mathrm{MC}}$	$T_{c,\mathrm{TRG}}$
3	1.81632(6)	1.8166(5) [18]
4	1.59082(3)	-
5	1.45045(2)	-
6	1.35242(3)	-
7	1.27889(3)	-
8	1.22101(5)	-
9	1.17387(5)	-
10	1.13446(4)	1.1330(20)
11	-	1.0998(10)
12	-	1.0705(10)
13	-	1.0450(10)
14	-	1.0226(10)
15	-	1.0016(10)
16	-	0.9834(10)
17	-	0.9669(10)
18	-	0.9508(10)
19	-	0.9371(10)
20	-	0.9247(10)

New results!

Summary

Tensor network methods have potential to assist in various interesting problems in Physics. On one hand, it can efficiently reproduce the ground state of several quantum systems with MPS and PEPS while on the other hand it can also describe real-space RG in various dimensions and can help us in understanding spin models, complex action systems, gauge theories etc. It is indeed a very exciting approach to Wilson's 4th aspect of RG!

Thank you