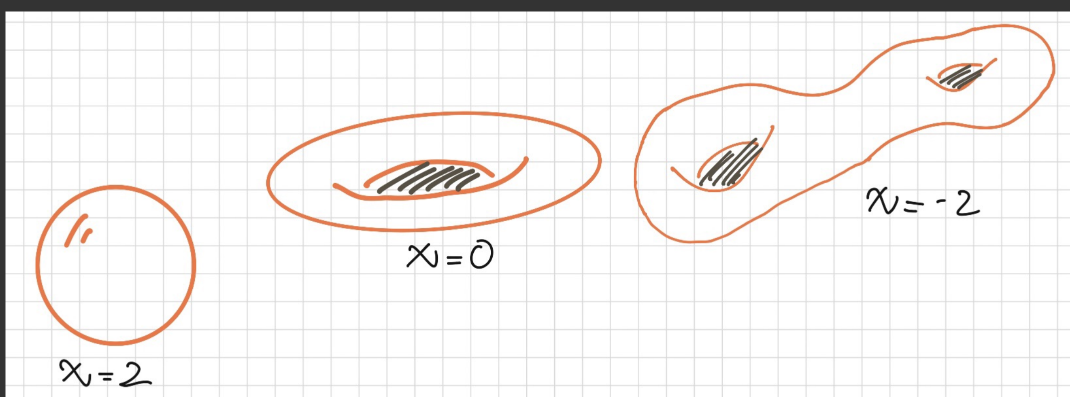
Solving Matrix Models

at large and finite N.



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Raghav Govind Jha

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

rjha1@perimeterinstitute.ca

Lectures Plan

1) Large N limit of matrix models

2) One matrix model – Analytic & Numerical solution

3) Generalization to three matrix model chain

4) Holographic matrix models

Classic reference: 2d quantum gravity & Random Matrices
hep-th/9306153

Large N

The theory of strong interactions i.e., QCD has a gauge group SU(3), where SU stands for special unitary i.e, matrices which are unitary with determinant equal to one.

't Hooft (1974) proposed that in some cases, if we take SU(N) rather than SU(3), it becomes easier.

At first, it seems surprising that considering SU(N) with large N is any better than SU(3) but this is not true.

Usually the SU(N) Yang-Mills action is given by:

$$S = \frac{1}{4g^2}\int d^4x \, \mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right)$$

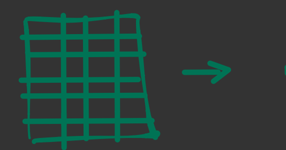
for large N limit, we consider $\lambda = g^2 N$ ('t Hooft coupling)

$$S = \frac{N}{4\lambda}\int d^4x \, \mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right)$$

Large N has some nice properties:

* Factorization: $\langle AB \rangle = \langle A \rangle \langle B \rangle + \mathcal{O}(1/N)$!

* Volume reduction: Eguchi-Kawai reduction!



(center symmetry intact)!!

* Role in holography: AdS/CFT conjecture! aka gauge/gravity duality

* Master field: Gauge field configurations dominated by single configuration!

$\hookrightarrow$ No progress in $d > 2$ yet!!

$\rightarrow$

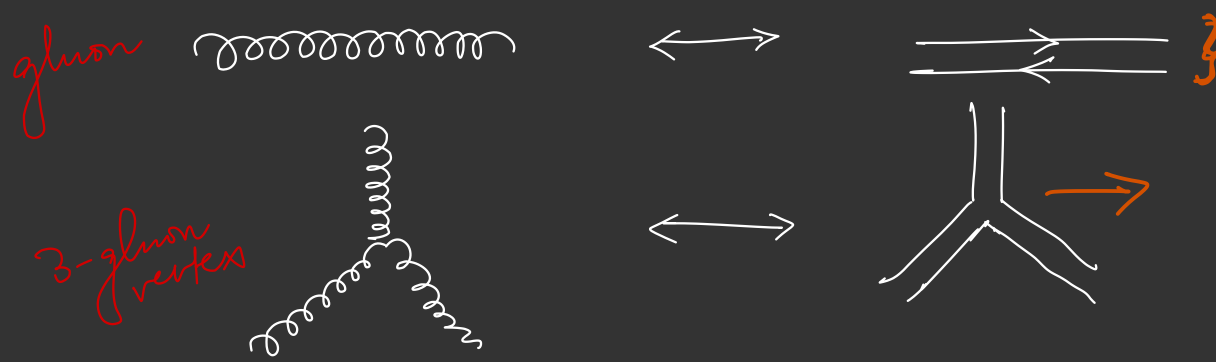
$$\mathcal{U}_\mu \mapsto e^{i\phi}\, \mathcal{U}_\mu$$

$$\phi \in \left\{ 0, \frac{2\pi}{N}, \dots \frac{2(N-1)\pi}{N} \right\} \quad \mathbb{Z}_N$$

Diagrams

One often uses the double line notation for example:

F.D		D.L.N



$E = 6, \quad V = F = 4$

Contribution $\sim N^{\chi} \lambda^{E-V}$

$\sim N^2 \lambda^2$

where $\chi = F + V - E$

Simplest large N models

Matrix models (aka Matrix integral)

$$Z = \int dM \, e^{-\mathrm{Tr}\, V(M)}$$

where M = hermitian/unitary matrices

This is known as 1MM (one matrix model)

For ex:

$$Z = \int dA \, dB \, e^{-\mathrm{Tr}\, V(A,B)}$$

is referred to as 2MM.

But a wide class of MM are not exactly solvable in the large N limit.

Fortunately, 1 MM (hermitian) is solvable.

Reference: Brezin, Itzykson, Zuber, Parisi - 1978 (BIPZ)

We'll not give the details but the basic idea is:

$$Z = \int dM \, e^{-\frac{N}{g^2} \operatorname{Tr} V(M)}$$

Jacobian

$$= \int \prod_i \frac{d\lambda_i}{2\pi} \, \underbrace{\Delta^2(\lambda)}_{\text{Vandermonde determinant}} \, e^{-\frac{N}{g^2} \sum_i V(\lambda_i)}$$

$$\boxed{\Delta^2(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)^2}$$

$$Z = \int \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} \, e^{N^2 S_{eff}(\lambda)} \quad \text{— } ①$$

where

$$S_{eff}(\lambda) = -\frac{1}{g^2 N^2} \sum_{i=1}^{N} V(\lambda_i) + \frac{2}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j|$$

In the $N \to \infty$, ① will be dominated by saddle point configuration that extremizes S_{eff}.

Varying one of the eigenvalues we get saddle point eq^n:

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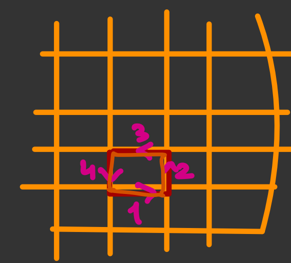
$$\frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}=\frac{V'(\lambda_i)}{g^2N} \qquad i=1\ldots N$$

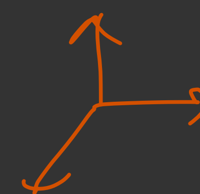
and then we can use complex analysis to pose this as Riemann-Hilbert problem and solve!

one-cut solution!!

UNITARY MATRIX MODELS ✓ Solvable

Classic example: Large N two-dimensional U(N) lattice gauge theory with Wilson action.





Wilson action

$$S(U) = \sum_p \frac{1}{g^2} \mathrm{Tr}\left(\prod_p U + h.c\right)$$

where $\prod_p U = U_1 U_2 U_3 U_4$

unitary

$$\underline{\underline{Z}} = \int \underline{\mathcal{D}U}\; e^{S}$$

→ Toeplitz matrix

$A_{ij} = f(i,j)$

$$= \mathrm{Det}\left[I_{j-k}\left(\frac{2N}{\lambda}\right)\right]$$

$j,k = 1 \ldots N$

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First studied by Gross, Witten, Wadia (~1980) and was shown to have large N phase transition of third-order at $\lambda = g^2 N = 2$.

EXERCISE : Compute $f = -\ln Z(N, \lambda)$ using Mathematica. Evaluate first and second derivative w.r.t λ !!

Solving Matrix models

analytical (1978) → now

Bootstrap (2020)

Monte Carlo (early 2000 onwards!) → major part of what follows!

• Exercise: Use the given `Mathematica` script and find the expected value for $\mathrm{Tr}(X^2)$ for one matrix model (1MM) with $g = 1.0$. This is the analytical result obtained in the planar limit using the well-known 'one-cut' method. Set up this problem for a fixed $N = 100$ and carry out the MC procedure. Compare the results.

Reproducing the exact result using Numerics

We use HMC algorithm to study the matrix models. → First introduced in Duane, Kennedy, Pendleton, Roweth 1987

Evolution using a time-reversible & volume preserving integrator (Ex: leapfrog)

Generate a series of configurations and estimate the average of a given observable as:

$$\langle A \rangle = \frac{1}{N} \sum_i A[\phi_i] \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)$$

HMC ALGORITHM

$$Z = \int \prod_x dp_x \prod_x d\phi_x \, e^{-H(\phi, p)}$$

and Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \sum_x p_x^2 + S(\varphi)$$

This is then evolved as per Hamilton's eqⁿ using a fictitious time 'τ' (also knows as MDTU)

$$\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{dp}{dt} = \frac{-\partial H}{\partial \phi}$$

$$\varphi = \varphi + p\,\delta t$$

$$p = p + dp = \left(-\frac{\partial H}{\partial \phi}\right) dt$$

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At the end of the evolution, an ACCEPT/REJECT test is done.

Summarize HMC :-

1) Choose momentum 'p' from $P(p_i) \propto e^{\frac{-p_i^2}{2}}$

2) Solve Hamilton's equation numerically for some time 'τ' where $(p, \phi) \longrightarrow (p', \phi')$

3) Accept proposed ϕ' with probability

$$P_{acc} = \min\left[1, e^{-\Delta H}\right]$$

where $\Delta H = H(\phi', p') - H(\phi, p)$

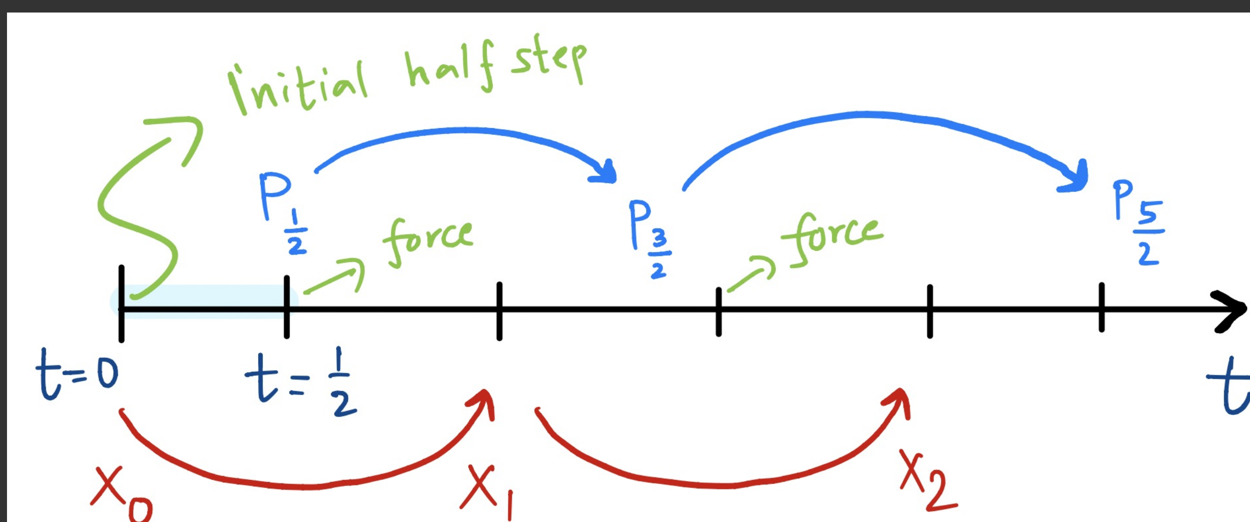
- $X_i(\delta\tau/2) = X_i(0) + P_i(0)\delta\tau/2$
- Now several inner steps where $n = 1 \cdots (N-1)$

$$P(n\delta\tau) = P((n-1)\delta\tau) - f_i((n-\tfrac{1}{2})\delta\tau)\delta\tau$$
$$X((n+1/2)\delta\tau) = X((n-1/2)\delta\tau) + P_i(n\delta\tau)\delta\tau$$

- $P(N\delta\tau) = P((N-1)\delta\tau) - f_i((N-\tfrac{1}{2})\delta\tau)\delta\tau$
- $X(N\delta\tau) = X((N-1/2)\delta\tau) + P_i(N\delta\tau)\frac{\delta\tau}{2}$

- Exercise:

LEAPFROG ALGORITHM in PYTHON } IMPLEMENT



Phase-Space Area

Leapfrog and even fancier algorithms (like Omelyan) both share property of preserving phase space area. These are properties of exact solution so it is nice to satisfy them in Numerics. It means that during an update $(\phi, p) \longmapsto (\phi', p')$ we conserve phase space measure.

like leapfrog!

- Exercise: Show that if a symplectic integrator is used, then $\langle e^{-\Delta H} \rangle = 1$ within errors.

Generating Momentum at Start

Since we are eventually interested in the large N limit of matrix models, we will be interested in generating N×N momentum matrices ?.

We will now discuss how to generate momentum matrices at the start of the leapfrog method taken from a Gaussian distribution. Suppose we have two numbers U and V taken from uniform distribution i.e., (0,1) and we want two random numbers with probability density function $p(X)$ and $p(Y)$ given by:

$$p(X) = \frac{1}{\sqrt{2\pi}} e^{-X^2/2} \tag{4.8}$$

and,

$$p(Y) = \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} \tag{4.9}$$

Since X and Y are independent sets:

$$p(X,Y) = p(X)p(Y) = \frac{1}{2\pi} e^{-R^2/2} = p(R,\Theta) \tag{4.10}$$

where $R = X^2 + Y^2$. This then means that we can identify the below:

$$U = \frac{\Theta}{2\pi} \tag{4.11}$$

and,

$$V = e^{-R^2/2} \implies R = \sqrt{-2\log(V)} \tag{4.12}$$

This then immediately means,

$$X = R\cos\Theta = \sqrt{-2\log(V)}\cos(2\pi U) \tag{4.13}$$

$$Y = R\sin\Theta = \sqrt{-2\log(V)}\sin(2\pi U) \tag{4.14}$$

EXERCISE

Implement this in Python !!

use random.uniform(0, 1)

and check that it indeed generates Gaussian distribution !!

1)

Random Hermitian matrices

```
def random_hermitian():

    tmp = np.zeros((NCOL, NCOL), dtype=complex)

    for i in range (NCOL):
        for j in range (i+1, NCOL):

            r1, r2 = box_muller()
            tmp[i][i] = complex(r1, 0.0)
            tmp[i][j] = complex(r1, r2)/math.sqrt(2)
            tmp[j][i] = complex(r1, -r2)/math.sqrt(2)

    return tmp
```

$$S = \mathrm{Tr}\left[\frac{X^2}{2} + \frac{g X^4}{4}\right] \rightarrow \text{Hermitian matrix model..} \quad \Bigg\} \; X = X^\dagger$$

$\hookrightarrow$ implement in Python.

Compute forces $f = -\dfrac{\partial S}{\partial X} = -\dfrac{\partial S}{\partial X^\dagger}$

Plot the expectation value of $\mathrm{Tr}(X^2)$ vs. MDTU for $g = 1$ and compare to exact value.

we'll use shorthand:

$$t_q = \frac{\mathrm{Tr}(X^q)}{N}$$

1 MM is SPECIAL

• Exercise: In a previous exercise, we computed $\mathrm{Tr}(X^2)$ for the one matrix model case. Why did we not consider higher moments of X? *Hint:* Use loop equations

$$\left\langle \mathrm{Tr} M^k V'(M) \right\rangle = \sum_{l=0}^{k-1} \langle \mathrm{Tr} M^l \rangle \langle \mathrm{Tr} M^{k-l-1} \rangle$$

and argue that the results we obtained are consistent.

Exercise: Find t_6 in terms of t_2 and check !!

END OF LECTURE 1

General Matrix Models

$$Z_p(g,c,k) = \int DM_1 DM_2 \cdots DM_p \exp \mathrm{Tr}\left(\sum_{i=1}^{p} -M_i^2 - g M_i^4 + c \sum_{i=1}^{p-1} M_i M_{i+1} + k M_p M_1 \right)$$

First considered in Chadha, Mahoux, Mehta (1981)
with $k=0$

$k=0$ → Solvable

$k \neq 0$ → Not solvable

Solvable ⟷ Not solvable: Monte Carlo numerics.

For simplicity consider $p=3$ now.

3 MATRIX MODEL

- Exercise: Set up the action for this three-matrix model chain problem and implement the HMC algorithm. Explore the limit when the chain is open i.e., $\kappa = 0$.

Consider $\mathrm{Tr}(X_{1,2,3}^2)$ for $g = 1$ and $c = k = 0.5$

MORE COMPLICATED MODELS

Until now, we have only considered interactions between matrices of form : $\mathrm{tr}(AB)$ which we call quadratic in matrices. Now, we consider quartic interactions of the form: $\mathrm{tr}([A,B]^2)$ for two-matrix model.

This will be important ingredient when we consider models which have holographic interpretation.

Hoppe's Matrix Model

$$Z = \int DX\,DY\, e^{-N\,\mathrm{Tr}\left(x^2+y^2-g^2[x,y]^2\right)}$$

Compute $\mathrm{Tr}([X,Y])^2$ vs. g and check that it is zero as $g\to\infty$.

References: hep-th/9810035 [Kazakov, Kostov, Nekrasov] and citations to this.

10 MM CLASS OF MODELS

IKKT model :

$\hookrightarrow$ hep-th/9612115

$$Z = \int dA\, d\psi\, e^{-S}$$

where
$$S = \underbrace{\frac{1}{4}\,\mathrm{Tr}\,[A_\mu, A_\nu]^2}_{S_{bos}} \underbrace{- \frac{1}{2}\,\mathrm{Tr}\;\bar{\psi}\Gamma^\mu[A_\mu, \psi]}_{S_{fermion}}$$

$\mu, \nu = 1 \dots 10$

generally $1 \dots d$

EXACT RESULT : SCHWINGER - DYSON IDENTITY

$$\boxed{\frac{\langle S \rangle}{N^2-1} = \frac{d}{4}}$$

Let's consider bosonic sector of the model and perform HMC and compare to the above result.

Estimate autocorrelation time of the average action computed !

Errors : 1210.3781

SUMMARY

We studied a wide range of matrix models which are used in different areas of Physics ranging from Nuclear Physics, Statistical Mechanics, Supersymmetry, Holography and Quantum Gravity.

We considered some solvable analytically models and checked that Monte Carlo reproduces those results. Then we moved to more complicated ones where exact treatment is not possible.

END OF LECTURE 2