New tool *for* old problems - Tensor network approach to spin models *and* gauge theories

1901.11443 PRD 99, 114507 (2019)
2004.06314 J. Stat. Mech. (2020) 083203
and work in progress!

Raghav Govind Jha (Perimeter)
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PERIMETER PI INSTITUTE FOR THEORETICAL PHYSICS

Outline

TRG

- ★ Introduction to tensor networks and Tensor Renormalization Group
- ★ Applications to 2d Ising model & Classical XY model
- ★ TRG for non-Abelian SU(2) gauge/Higgs model in two dimensions.
- ★ Future directions

Introduction

Different renormalization group (RG) methods have been introduced over the past 5-6 decades:

- Kadanoff's spin blocking RG [1966] & Wilson's Numerical RG [1970s]
- Density Matrix Renormalization Group (DMRG) [White, 1992]
  (DMRG is a refined extension to above approach and is well-suited to all 1d systems not only restricted to impurity problems such as Kondo problem.)
- Tensor Renormalization Group [Levin and Nave, 2007]
  (Like DMRG breaks down at criticality.)
- Tensor Network Renormalization (TNR) [Vidal and Evenbly, 2015]
  (TNR is an extension of TRG which qualitatively improves TRG behaviour for systems at criticality and can be used to generate MERA tensor networks)

Motivations

- Formulating in terms of tensors can enable us to study systems where the usual Monte Carlo (MC) methods fail (sign problem!). In addition, the partition function is directly accessible in the thermodynamic limit unlike MC methods enabling direct study of thermodynamic quantities.

- Provides an arena for studying lower-dimensional critical and gapped systems faster and more efficiently than any other numerical method available.

- Has recently been understood to play an important role in understanding the AdS/CFT (i.e. bulk physics from entangled quantum state at the boundary).

Basic idea

- Tensor networks approach belong to two categories: Lagrangian and Hamiltonian approaches. For ex:

  $|\Psi\rangle = \sum_{i_1,i_2,\cdots,i_N} C_{i_1\cdots i_N} |i_1 i_2 \cdots i_N\rangle$ as approximation to the ground state wave function of complicated many-body quantum system with local Hamiltonian. Ex: Matrix Product States (MPS) representation. Reduction to O(N) rather than O(d^N) coefficients.

  $Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})}$ to approximate in the Lagrangian formulation (like we consider in this talk later).

Basic idea

- As mentioned, for computing "Z", we have to identify the important states and keep them. The effectiveness of these methods depends on the nature of the system. A general random state has a volume scaling of EE while it is different for ground-state sector. For example: we can have gapped and critical (gapless) systems and they have following behaviour -

Hastings [2007]

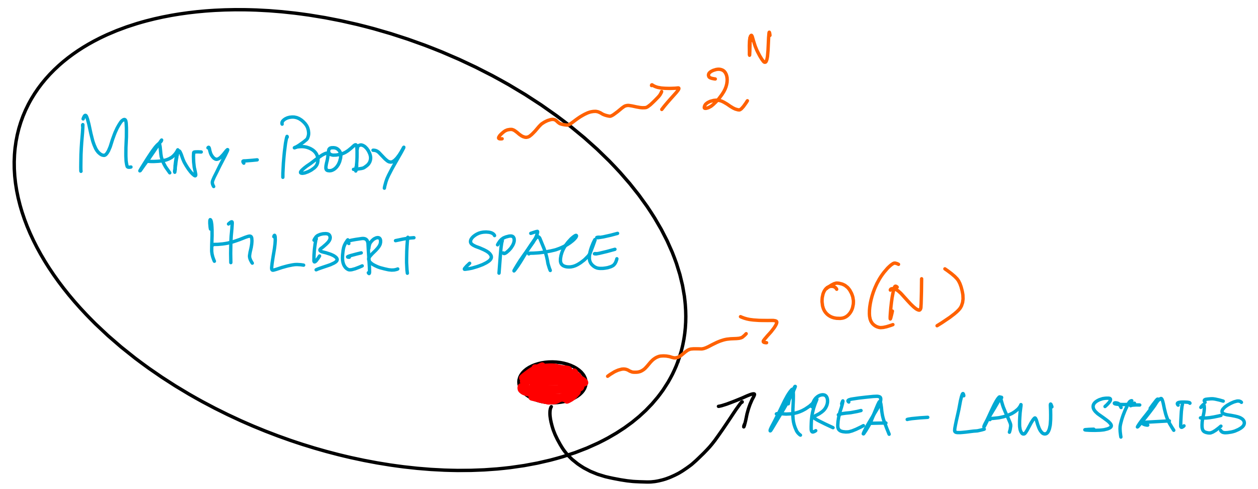
$\mathscr{H}$	$C(r)$	$S_{1+1}(A)$
Gapped	$\sim \exp[-r/\xi]$	$\sim L^0 \sim$ constant
Gapless/Critical	$\sim 1/(r)^q$	$\sim \ln(L_A)$

Area-law

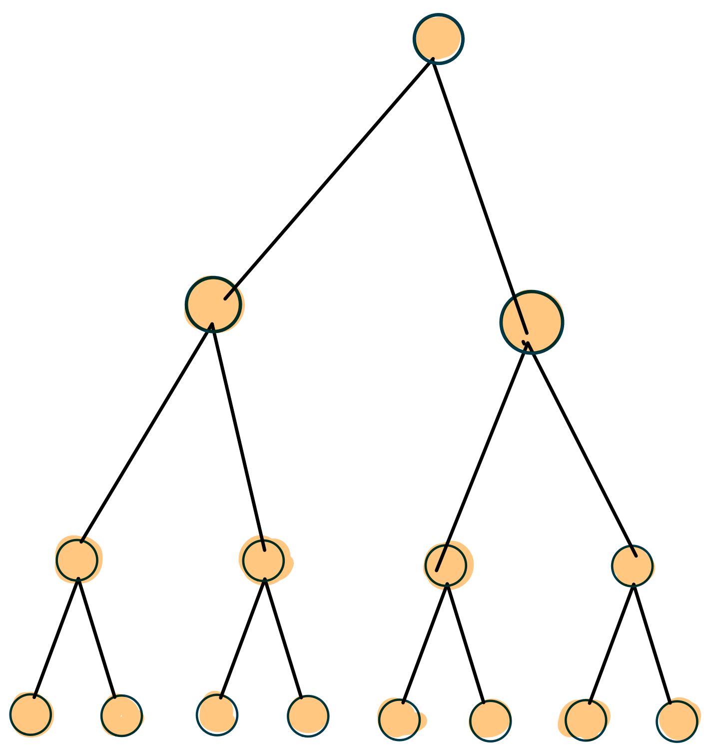
Same scaling as EE in AdS3

How to identify?

- Ground states are not "arbitrary" states in Hilbert space, it has special features. Some of these have been captured by studying the entanglement entropy (EE). The region of Hilbert space that obeys area-law scaling for the EE corresponds to a tiny corner (in red). Therefore, lot of progress have been made in many-body physics by computing EE and hence identifying important regions of Hilbert space.



TTN/MERA (Multi-scale Entanglement Ansatz)



TREE TENSOR NETWORK

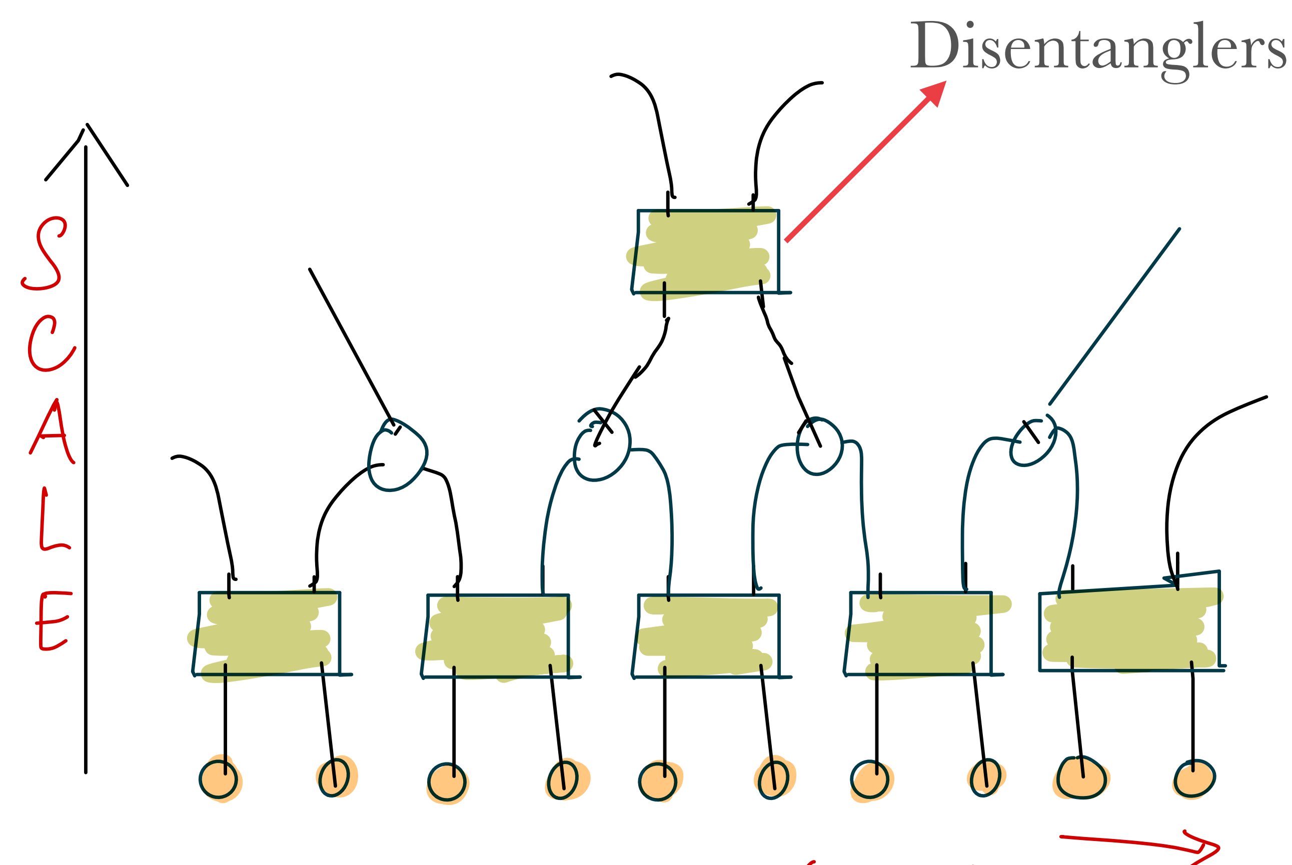
(gapped systems)

↓

finite correlation length

↓

Area-law

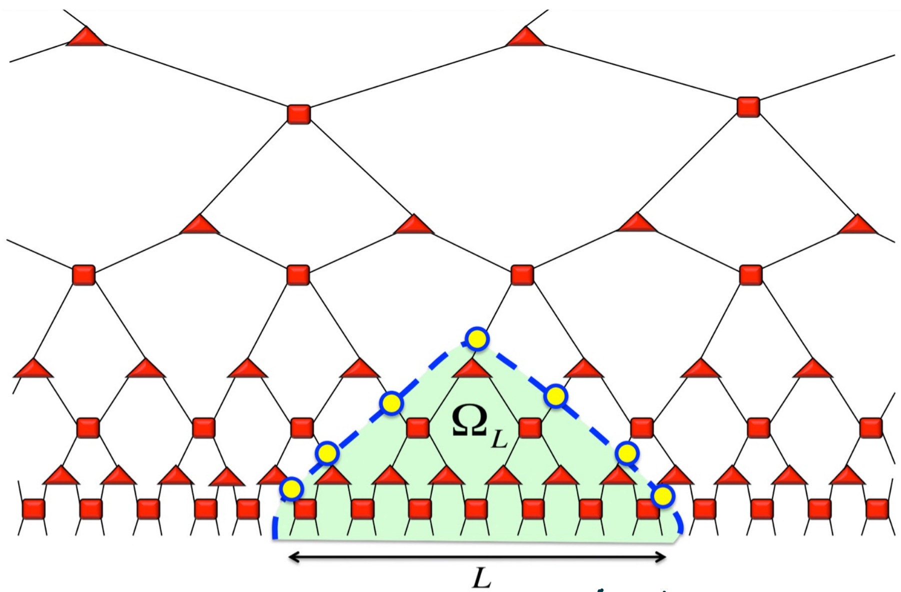


MERA (1-d)

(effecient for critical system)

S_A has logarithmic behaviour

MERA



arxiv 1812.04011

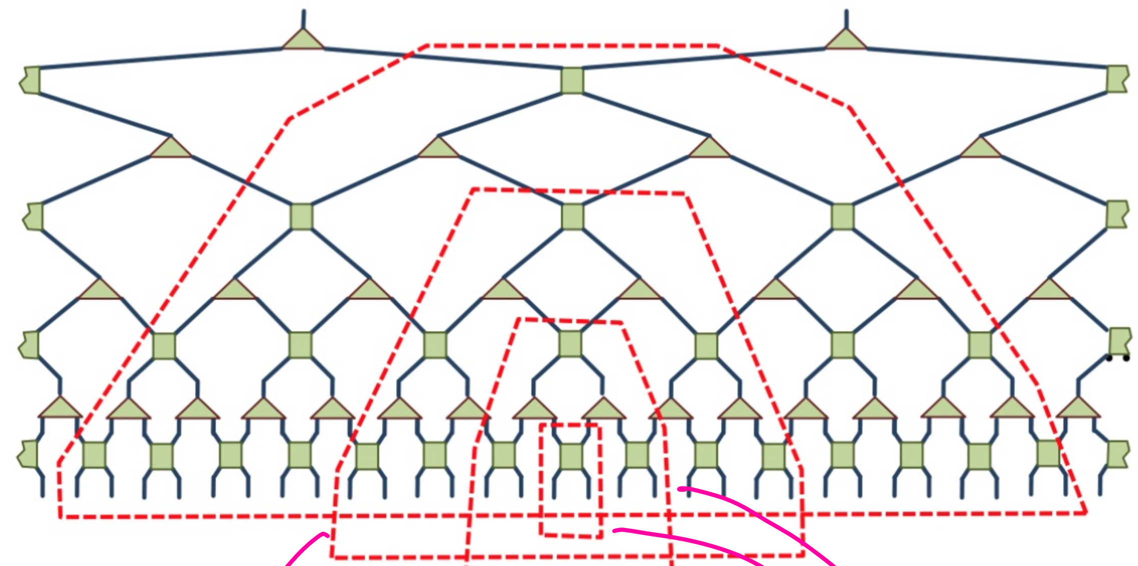


Figure Courtesy: G. Vidal

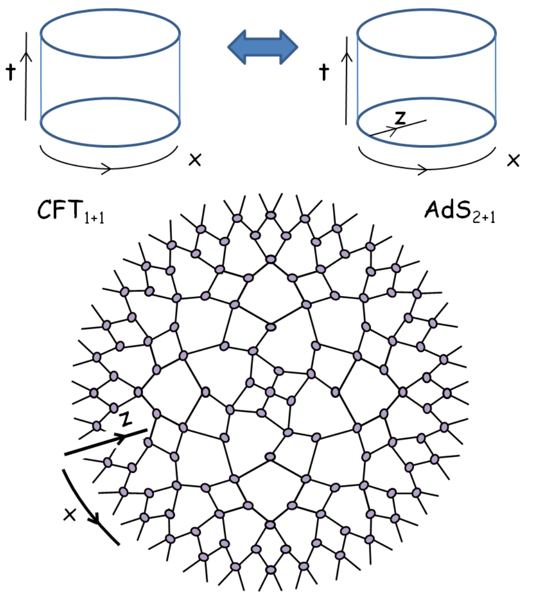
$\{14, 6\}$ $\{2, 2\}$ $\{6, 4\}$

$$\partial\Omega_L = O(\log L)$$

disentanglers: removing short-range entanglement $V \otimes V \to V \otimes V$

isometries: mapping block of sites to one $V \otimes V \to V$

What does MERA capture?



- Swingle (0905.1317) — MERA on the real line would describes a time slice of the Poincare patch of AdS3, which corresponds to the hyperbolic plane. So, MERA is the lattice realisation of AdS/CFT! (* Hint from Ryu-Takayanagi formula matching EE*)

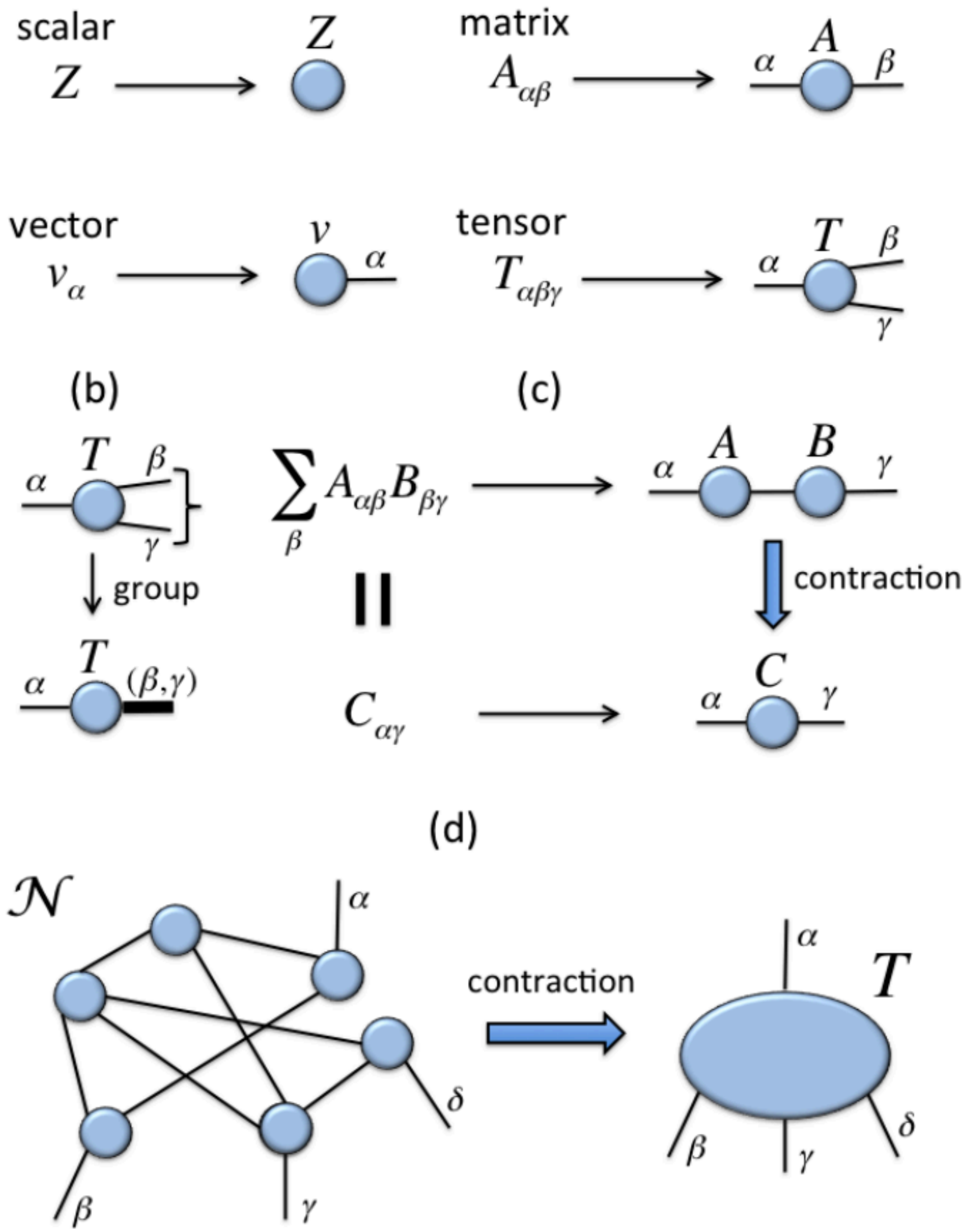
- Beny and others (1110.4872) — MERA on the real line should be interpreted instead as a Poincare patch of dS spacetime

- Vidal & Milsted (1812.00529) argued that MERA on the real line would describe light sheet geometry.

This talk..

- There exists several algorithms for gapped systems which work well for models we want to discuss in this talk. So we just have a simple tree tensor network (TTN) and no disentanglers. We will use a specific algorithm known as HOTRG which we now describe.

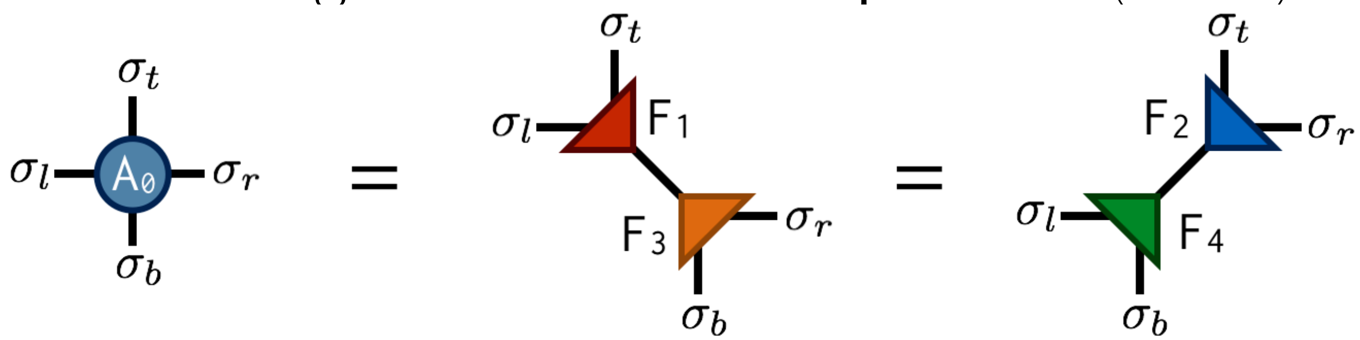
Notation



1112.4101

Basic idea - TRG (in 2d)

Assuming that the system is represented by a building block i.e. rank-4 tensor given by A_0. The first move is to do singular value decomposition (SVD) of this as shown below.



This is shown in steps as

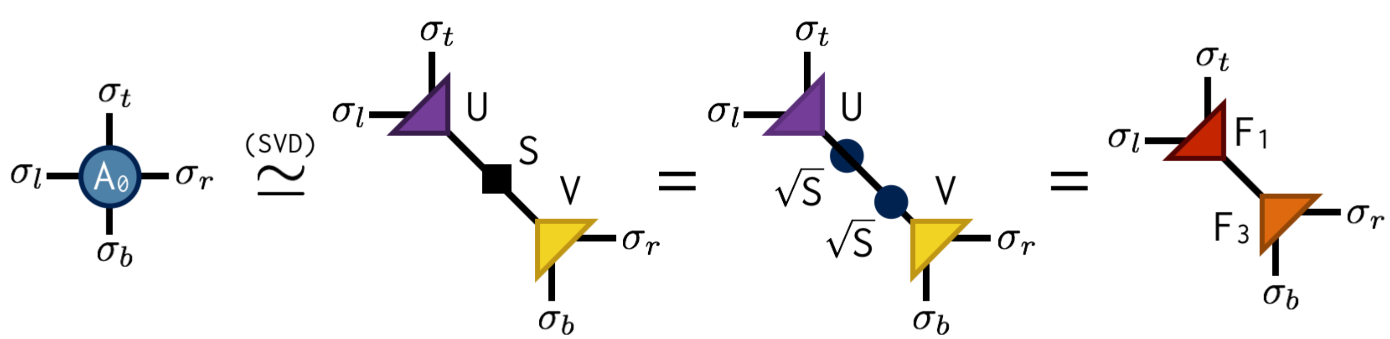


Figure: http://tensornetwork.org/trg/

Basic idea - TRG (2d)

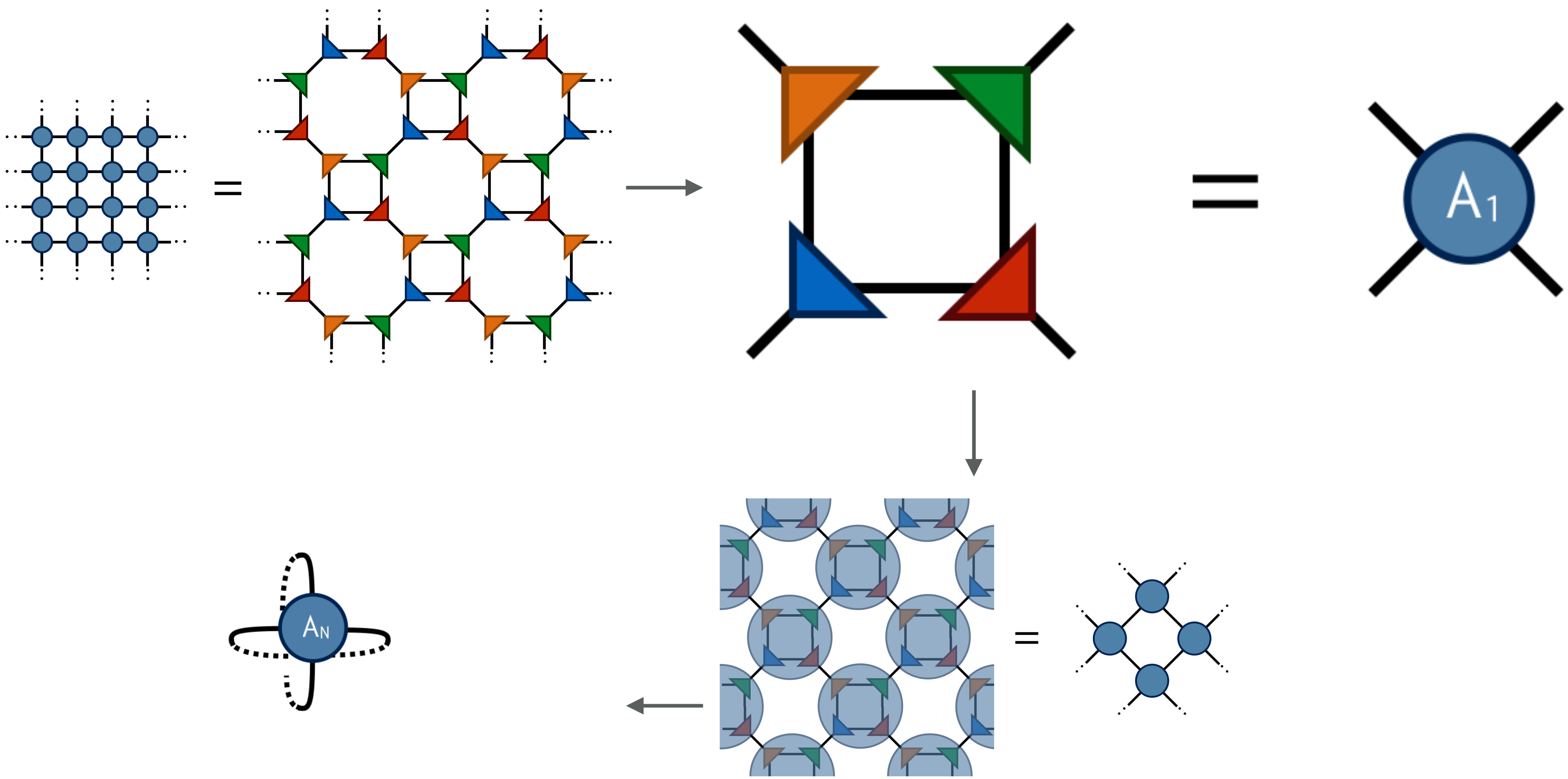


Figure: http://tensornetwork.org/trg/

HOTRG (Higher-order TRG)

A refined real space coarse graining method similar in spirit to TRG but employs higher-order SVD (HOSVD) to minimise the errors due to truncation. First introduced in 1201.1144 and is successfully applied to statistical systems in d = 2, 3, and recently in 4. Performs better than naive TRG for critical systems. Less complex than the TNR methods (best for critical systems!)

Coarse-graining renormalization by higher-order singular value decomposition

Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, T. Xiang

We propose a novel coarse graining tensor renormalization group method based on the higher-order singular value decomposition. This method provides an accurate but low computational cost technique for studying both classical and quantum lattice models in two- or three-dimensions. We have demonstrated this method using the Ising model on the square and cubic lattices. By keeping up to 16 bond basis states, we obtain by far the most accurate numerical renormalization group results for the 3D Ising model. We have also applied the method to study the ground state as well as finite temperature properties for the two-dimensional quantum transverse Ising model and obtain the results which are consistent with published data.

Example - 2d Ising [Square lattice]

Exactly solvable system with solution due to Onsager [1944], where the logarithm of the partition function is given by:

$$f(\beta) = -\frac{1}{\beta}\left(\ln(2) + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln\left[2\cosh^2(2\beta) - \sinh(2\beta)\cos(\phi_1) - \sinh(2\beta)\cos(\phi_2)\right] d\phi_1 d\phi_2 \right)$$

And has singularity (phase transition) at:

$$T_c = \frac{2}{\ln(1+\sqrt{2})} = 2.26918531421 \implies \beta_c \approx 0.440687$$

We apply HOTRG to this system and match to known analytical results as a check.

Example - 2d Ising [Square lattice]

The fundamental tensor (first step in the tensor computations) can be written down as:

$$T_{abcd} = W_{ia}W_{ib}W_{ic}W_{id}$$

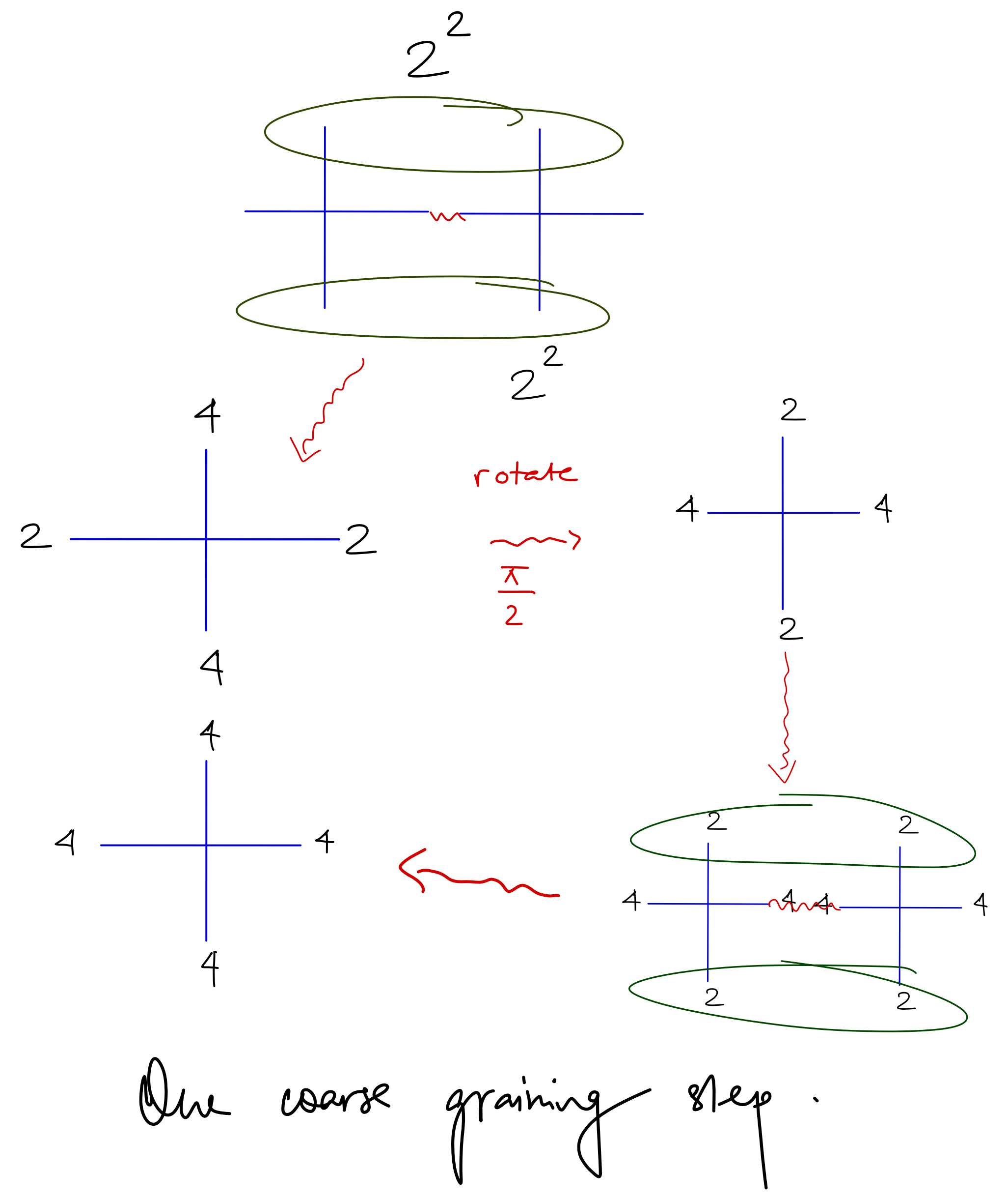
with W given by a 2×2 matrix:

$$W_{ia} = \begin{bmatrix} \sqrt{\cosh(\beta)} & \sqrt{\sinh(\beta)} \\ \sqrt{\cosh(\beta)} & -\sqrt{\sinh(\beta)} \end{bmatrix}$$

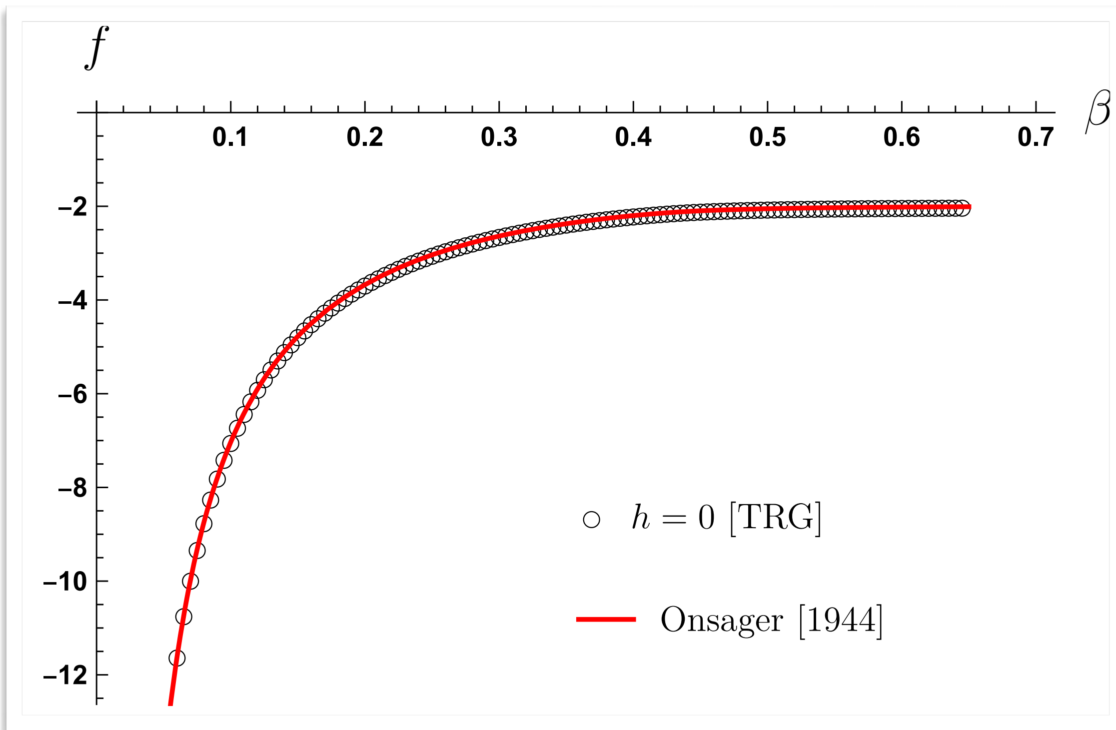
This simple spin model case does not require any initial truncation of the tensor. W can also be modified to include h to study model in magnetic field.

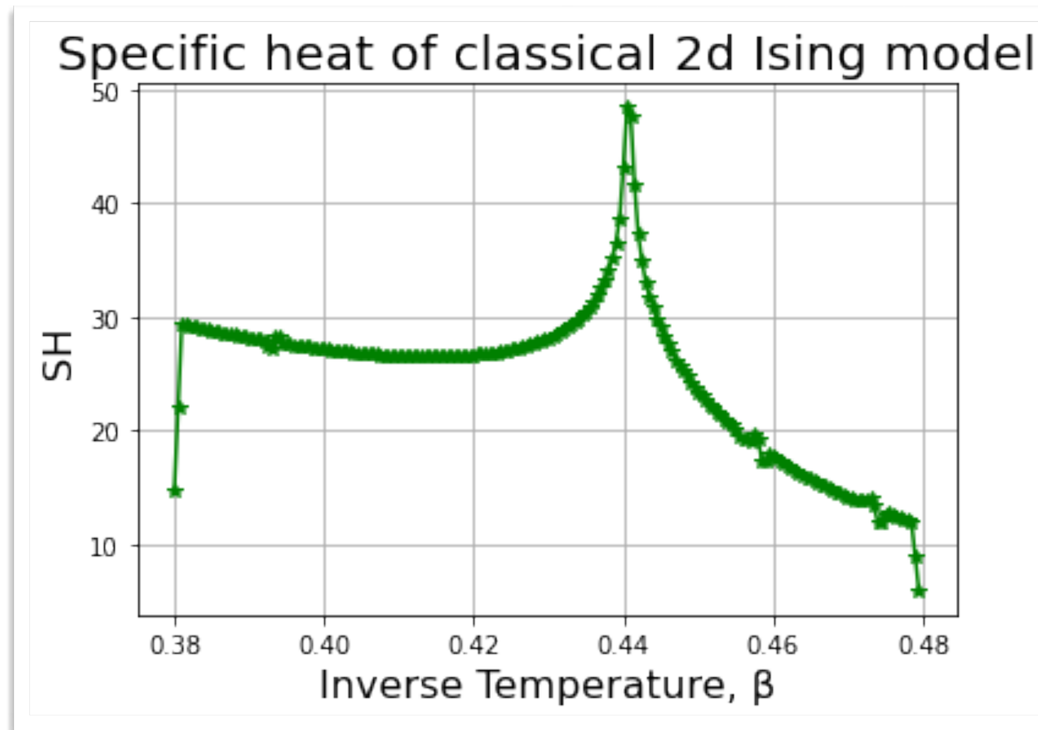
Example - 2d Ising [Square lattice]

We can carry out the first two iterations of coarse-graining exactly (first shown below), however we truncate to $\chi = 18$ from the third iteration onwards.



State-of-the-art numerical result





- Time: 20-25 seconds on a modern laptop (χ = 18)
- For Python notebook check: https://github.com/rgjha

Additional remark

Tensor network (TN) approach gives direct access to the canonical partition function unlike Monte Carlo methods. This has several advantages. In cases where exact form of Z is unknown, an educated guess could have been made about the closed form of Z (or its logarithm) comparing the results from TRG in the thermodynamic limit (i.e. on a lattice of size $2^{50} \times 2^{50}$).

This has also been useful for exploring the applications of tensor networks in case of supersymmetric gauge theories since Z under some appropriate boundary conditions gives well-known 'Witten-index'. For ex: see 1801. 04183 which studies two-dimensional lattice $\mathcal{N} = 1$ Wess–Zumino model using TNs. This is more involved that what we discuss in this talk because of "fermions". But there does exists tensor networks methods for them as well.

Classical XY model

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Simplest spin model with continuous symmetry O(2) [h=0] in two dimensions. The nearest neighbour Hamiltonian is given by:

$$\mathcal{H} = -J\sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) - h\sum_i \cos\theta_i$$

In order to construct the tensor representation, we decompose the Boltzmann weight (for say h=0) using Jacobi-Anger expansion as:

$$\exp\left(\beta\cos(\theta_i - \theta_j)\right) = I_0(\beta) + \sum_{\nu=-\infty,\neq 0}^{\infty} I_\nu(\beta)e^{i\nu(\theta_i - \theta_j)}$$

where, I_ν is the modified Bessel function of first kind.

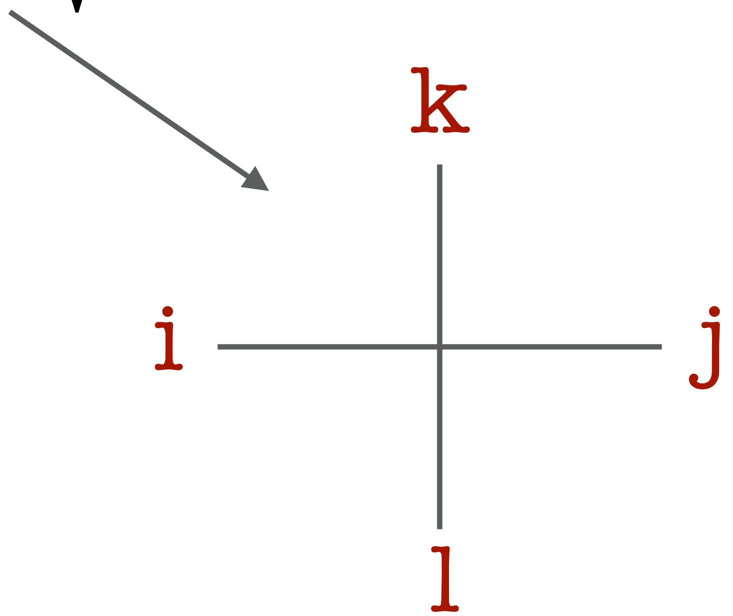
Classical XY model

The partition function can then be written as:

$$Z = \int \prod_i d\theta_i \prod_{\nu_{ij},\mu_i} I_{\nu_{ij}}(\beta) I_{\mu_i}(\beta h) e^{i\nu_{ij}(\theta_i - \theta_j) + i\mu_i\theta_i}$$

By integrating over $d\theta_i$, we obtain the initial tensor for XY model

$$T_{ijkl} = \sqrt{I_i(\beta)I_j(\beta)I_k(\beta)I_l(\beta)}\, I_{i+k-j-l}(\beta h)$$



Truncation step in TRG

Almost all TRG computation needs truncation of the states.

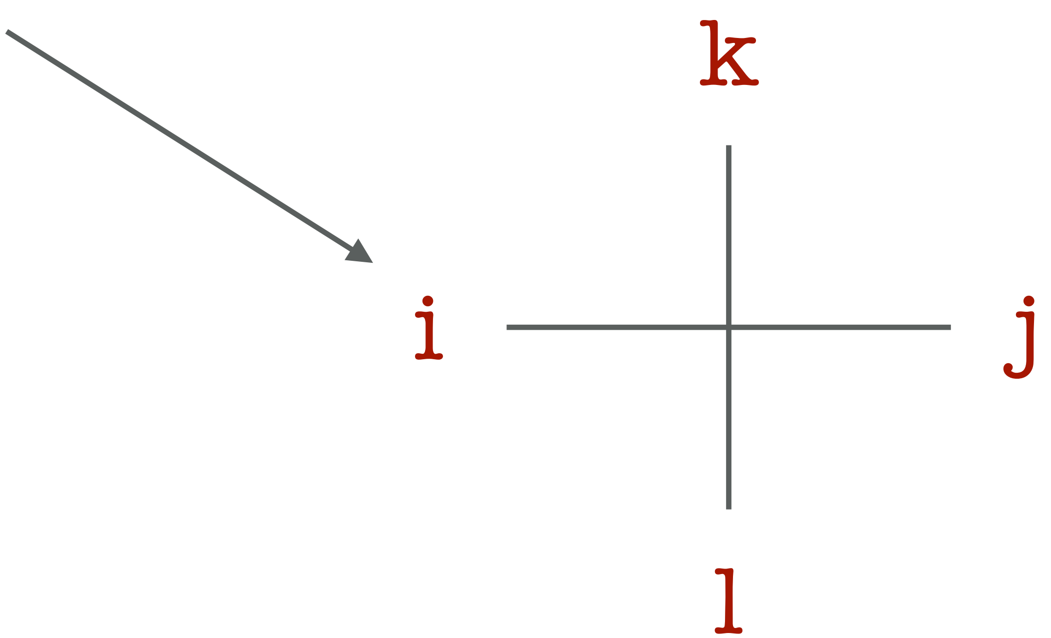
For ex: the indices ν_{ij}, ideally runs from $-\infty$ to ∞. For numerical implementation, we have to truncate the indices such that it only runs in integer steps from $-\chi/2$ to $\chi/2$ passing through 0.

Therefore, in the diagram below i, j, k, l runs from -26 to 26 with $\chi = 53$

$$T_{ijkl} = \sqrt{I_i(\beta)I_j(\beta)I_k(\beta)I_l(\beta)}I_{i+k-j-l}(\beta h)$$

Needs about ~170 Gb of memory since we need to

at least store at least χ^6 elements!



Classical XY model

This then leads to the partition function as:

$$Z = \mathrm{tTr} \prod \mathrm{T_{ijkl}}$$

from which the magnetisation can be computed as:

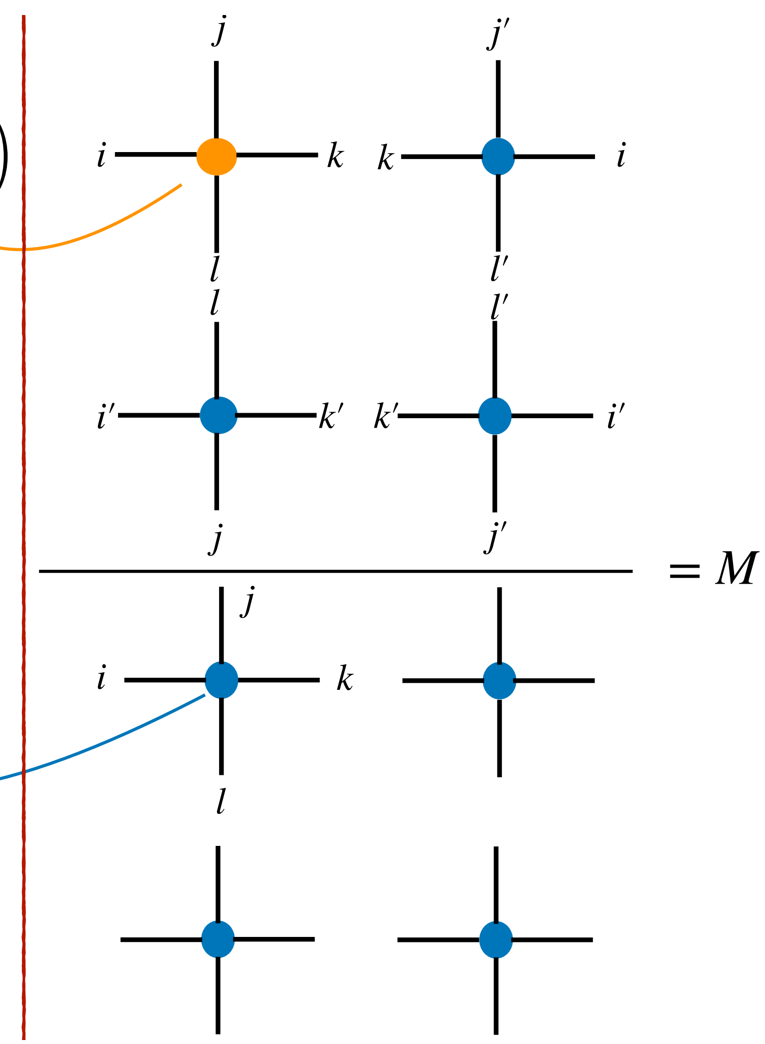
$$M = \frac{-\partial F}{\partial h} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial h} = \mathrm{tTr}\left(\sqrt{I_i(\beta)I_j(\beta)I_k(\beta)I_l(\beta)}\frac{I_{i+k-j-l-1}(\beta h) + I_{i+k-j-l+1}(\beta h)}{2}\right).$$

Continued..

The way we compute magnetisation using tensor networks is shown below:

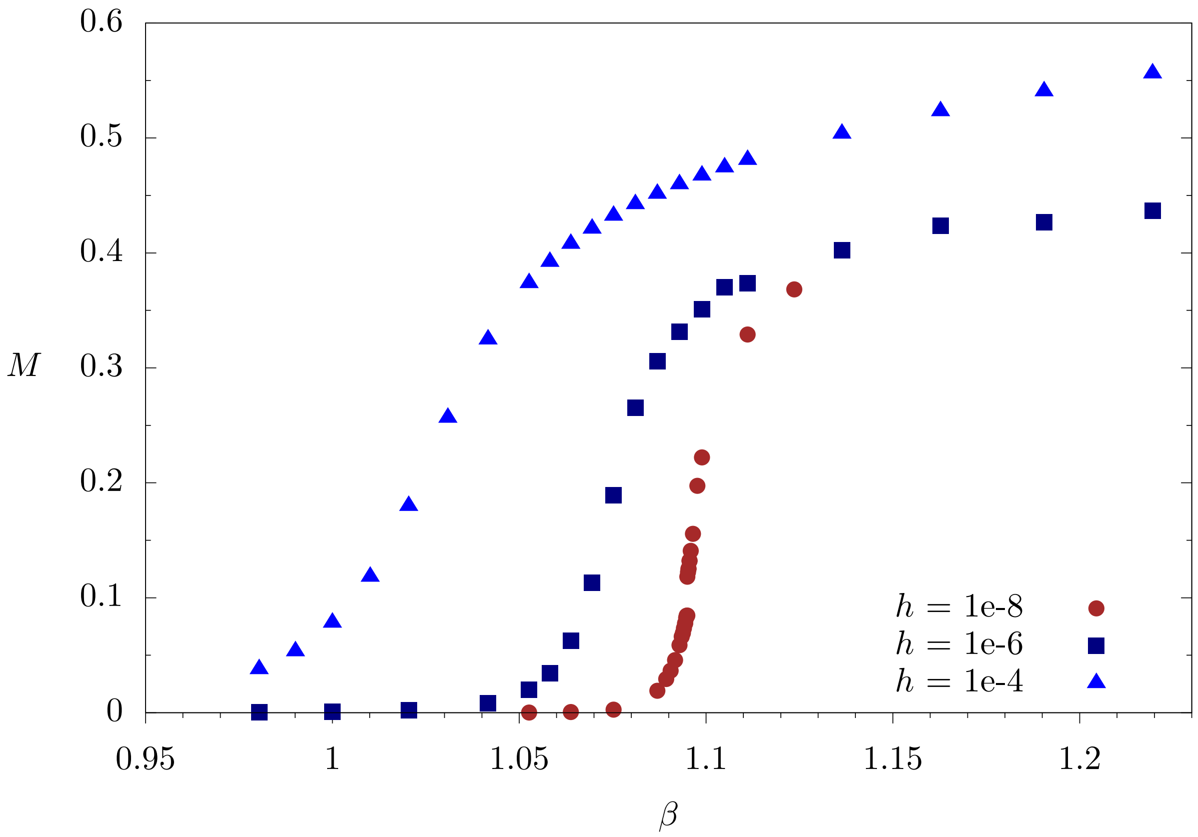
$$\tilde{T}_{ikjl} = \sqrt{I_i(\beta)I_k(\beta)I_j(\beta)I_l(\beta)}\left(\frac{I_{i+j-k-l-1}(\beta h)+I_{i+j-k-l+1}(\beta h)}{2}\right)$$

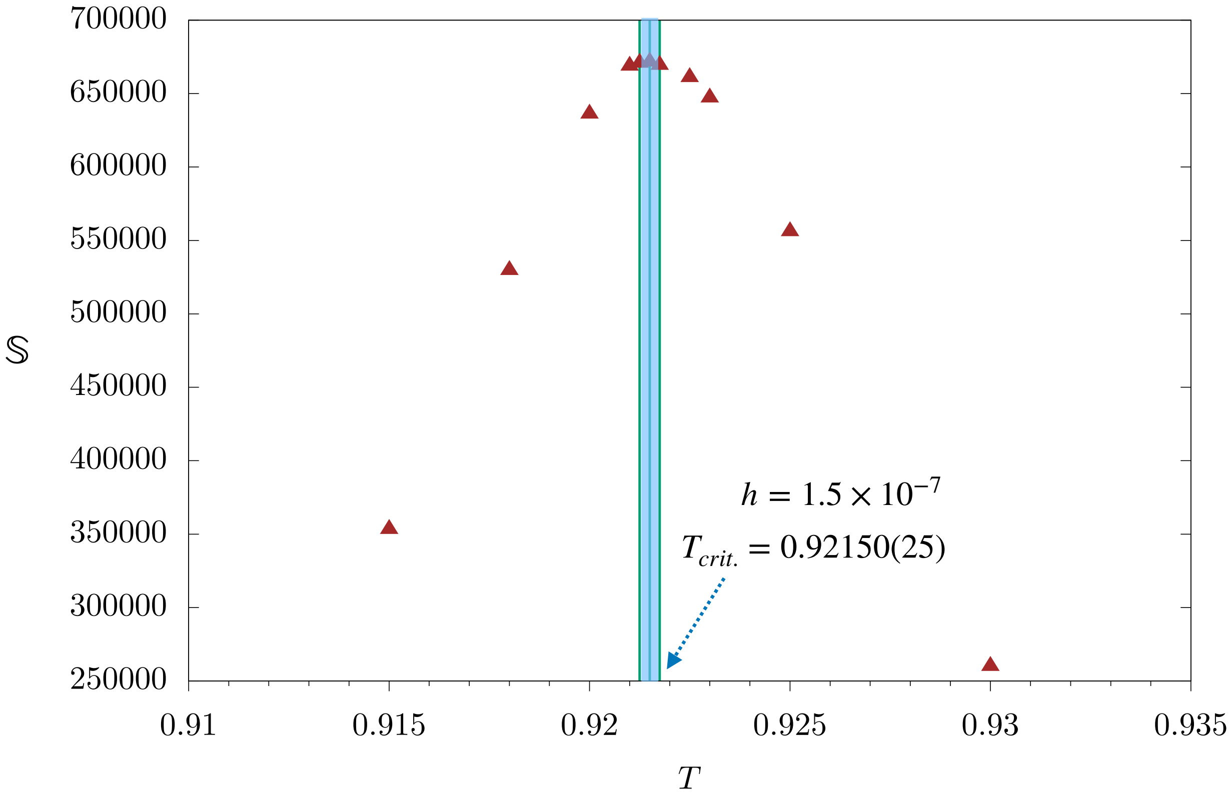
$$T_{ikjl} = \sqrt{I_i(\beta)I_k(\beta)I_j(\beta)I_l(\beta)}\,I_{i+j-k-l}(\beta h)$$



Orange blob is the impure or 'observable' tensor. The denominator is just the contracted usual network which gives "Z".

Taking derivative of magnetisation for small variation in magnetic field, we compute the susceptibility (right) for range of h and take the limit of vanishing field.





Taking the limit $h \to 0$, we obtain the critical temperature of 0.89290(5) which is consistent with the most precise MC results available in literature. This result can be further improved using larger bond dimension χ. It is known that in XY model correlation length increases exponentially as we approach $T = T_c$. Already at $T \approx 0.95$, the correlation length as noted in 1907.04576 is more than thousand lattice sites. Precise MC results used a square lattice of size $2^{16} \times 2^{16}$ while we used a lattice $2^{50} \times 2^{50}$.

Method	Year	System Size	T_{critical}	
Monte Carlo [21]	1992	$2^9 \times 2^9$	0.89400(500)	
HTE [22]	1993	–	0.89440(250)	
Monte Carlo [23]	1995	$2^8 \times 2^8$	0.89213(10)	
Monte Carlo [24]	2005	$2^{11} \times 2^{11}$	0.89294(8)	
HTE [25]	2011	–	0.89286(8)	
Monte Carlo [15]	2012	$2^{16} \times 2^{16}$	0.89289(5)	
Monte Carlo [26]	2013	$2^9 \times 2^9$	0.89350(10)	
Higher-order TRG [7]	2013	$2^{40} \times 2^{40}$	0.89210(190)	TN
Uniform MPS [8]	2019	–	0.89300(10)	TN
Higher-order TRG [This work]	2020	$2^{50} \times 2^{50}$	0.89290(5)	TN

2d non-Abelian gauge Higgs (NAGH) model

A Bazavov, S Catterall, RGJ, J Unmuth-Yockey, Phys. Rev. D 99, 114507 (2019), **1901.11443**

The lattice action for SU(2) is given by:

$$S = -\frac{\beta}{2}\mathrm{Re\ Tr}\square - \frac{\kappa}{2}\ \mathrm{Re\ TrU}$$

where β is the gauge coupling and κ is the matter coupling. We have fixed to unitary gauge as used in several previous works by Greensite et al. [Also Osterwalder-Seiler-Fradkin-Shenker].

We expand the Boltzmann weights in terms of characters (called character expansion). Writing, $S = S_g + S_\kappa$, we have for gauge piece:

$$e^{-S_g} = \prod_x \sum_r F_r(\beta)\chi^r(UUU^\dagger U^\dagger)$$

Link (A) and Plaquette (B) tensors

1901.11443

$$A_{(r_l m_{la} m_{lb})(r_r m_{ra} m_{rb})}(\kappa) = \frac{1}{d_{r_r}} \sum_{\sigma=|r_r-r_l|}^{r_r+r_l} F_\sigma(\kappa) C^{r_r m_{rb}}_{r_l m_{lb} \sigma (m_{rb}-m_{lb})} \times C^{r_r m_{ra}}_{r_l m_{la} \sigma (m_{rb}-m_{lb})} .$$

$$B_{(r_l m_{la} m_{lb})(r_r m_{ra} m_{rb})(r_a m_{al} m_{ar})(r_b m_{bl} m_{br})} = \begin{cases} F_r(\beta) \quad \delta_{m_{la},m_{al}} \delta_{m_{ar},m_{ra}} \delta_{m_{rb},m_{br}} \delta_{m_{bl},m_{lb}} & \text{if } r_l = r_r = r_a = r_b = r \\ 0 \qquad \text{else.} \end{cases}$$

With the knowledge of these two tensors, one can construct the fundamental tensor.

Representation of A and B tensors

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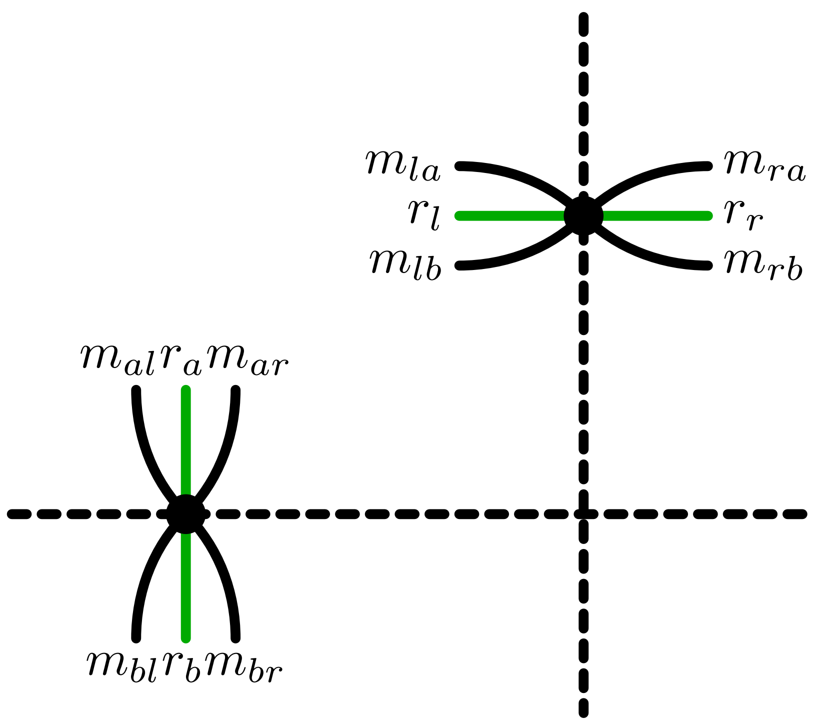
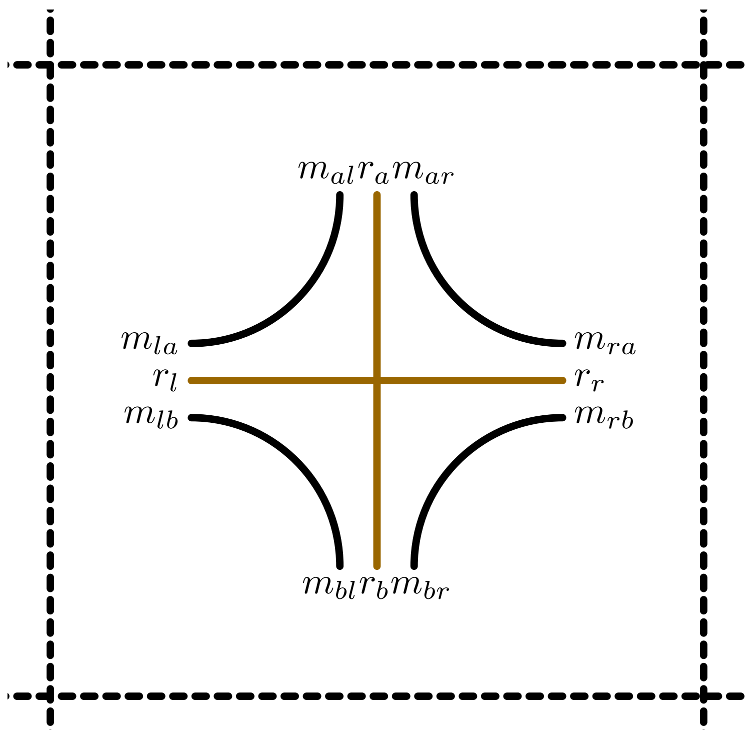


Rank-2 link tensor



Rank-4 plaquette tensor

Fundamental tensor (T)

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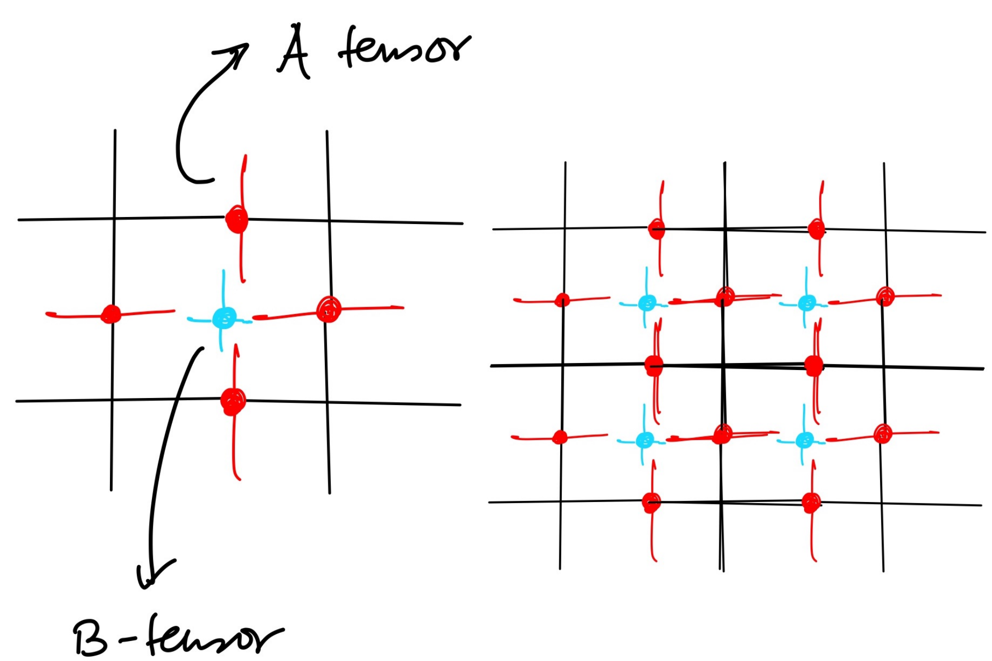
Using A and B tensors, we can construct the fundamental tensor of rank-4 (in 2d). And then combining several copies of this tensor we can make up the entire lattice. Note that we can just same A and B by exploiting the translational invariance of the lattice. If there were defects in the system, then one would need more such tensors.

$$T_{ijkl}(\beta, \kappa) = \sum_{\alpha.\beta.\gamma.\delta} B_{\alpha\beta\gamma\delta}(\beta) L_{\alpha i} L_{\beta j} L_{\gamma k} L_{\delta l}(\kappa), \quad \text{where} \quad A_{ij} = \sum_k L_{ik} L^T_{kj}$$



Cholesky decom.

Coarse graining the TN

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We use HOTRG discussed before to implement coarse graining with $\chi = 50$ where we have to obviously truncate over the "irreps" of SU(2) by considering upto $r_{\text{max.}} = 1$ which corresponds to initial tensor T of size 14^4. If we instead used $r_{\text{max.}} = 1/2$, then it would initial tensor T of size 5^4. This truncation over the "irreps" depends on the gauge theory and range of couplings one wants to study. Then, Z is given as usual by:

$$Z = \text{tTr} \prod \text{T}_{\text{ijkl}}$$

Some Observables

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- We compute Polyakov loop in a fixed representation of SU(2) by inserting an appropriate tensor given by:

$$\tilde{A}_{(r_l m_{la} m_{lb})(r_r m_{ra} m_{rb})ij}(\kappa) = \frac{1}{d_{r_r}} \sum_{r'=|\frac{1}{2}-r_l|}^{\frac{1}{2}+r_l} \sum_{\sigma=|r_r-r'|}^{r_r+r'} F_\sigma(\kappa) C^{r_r m_{rb}}_{r'(m_{lb}+i)\sigma(m_{rb}-m_{lb}-i)} C^{r_r m_{ra}}_{r'(m_{la}+j)\sigma(m_{rb}-m_{lb}-i)} C^{r'(m_{lb}+i)}_{r_l m_{lb} \frac{1}{2} i} C^{r'(m_{la}+j)}_{r_l m_{la} \frac{1}{2} j} .$$

- Correlation of Polyakov loop i.e. $\langle P(0) P^\dagger(r) \rangle = C(r)$ to identify Higgs-like or confining phases

Exact results for $\beta = 0$

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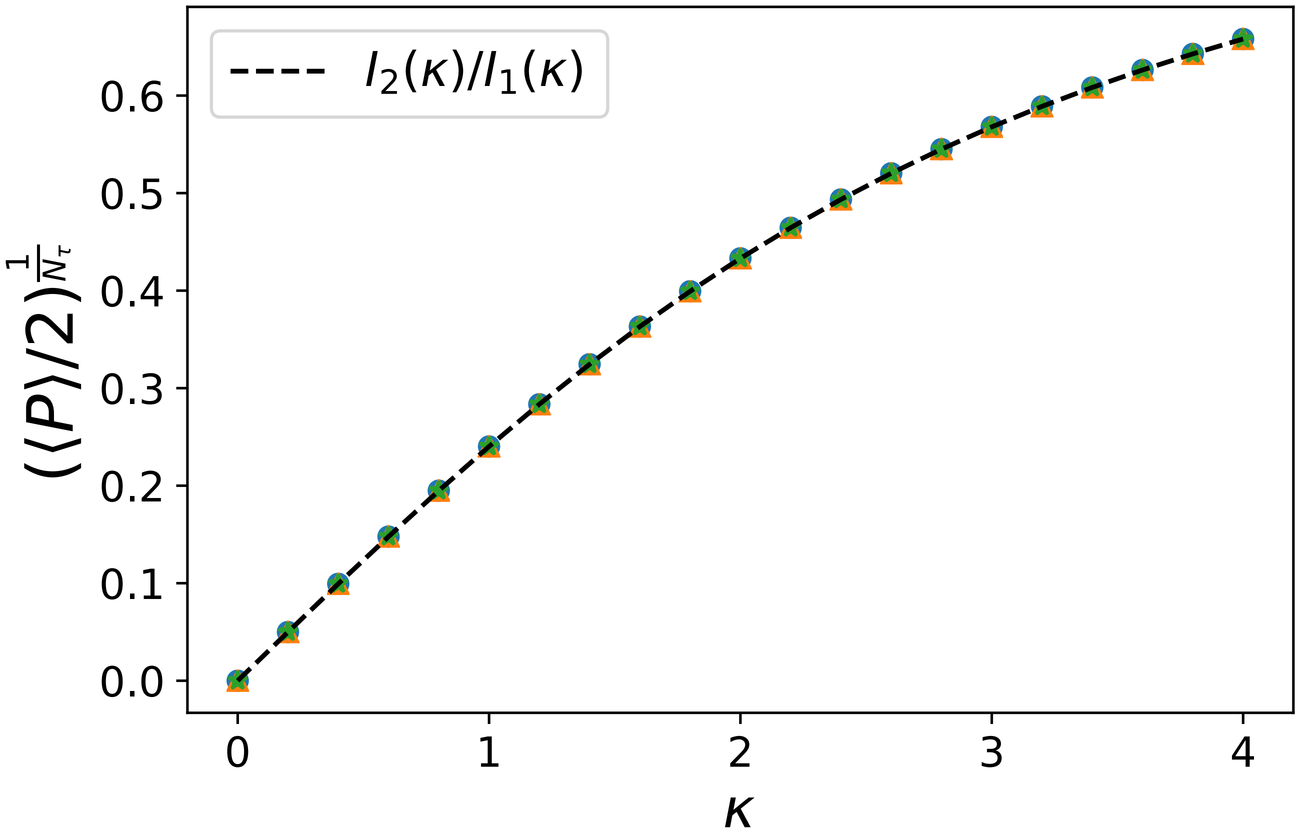
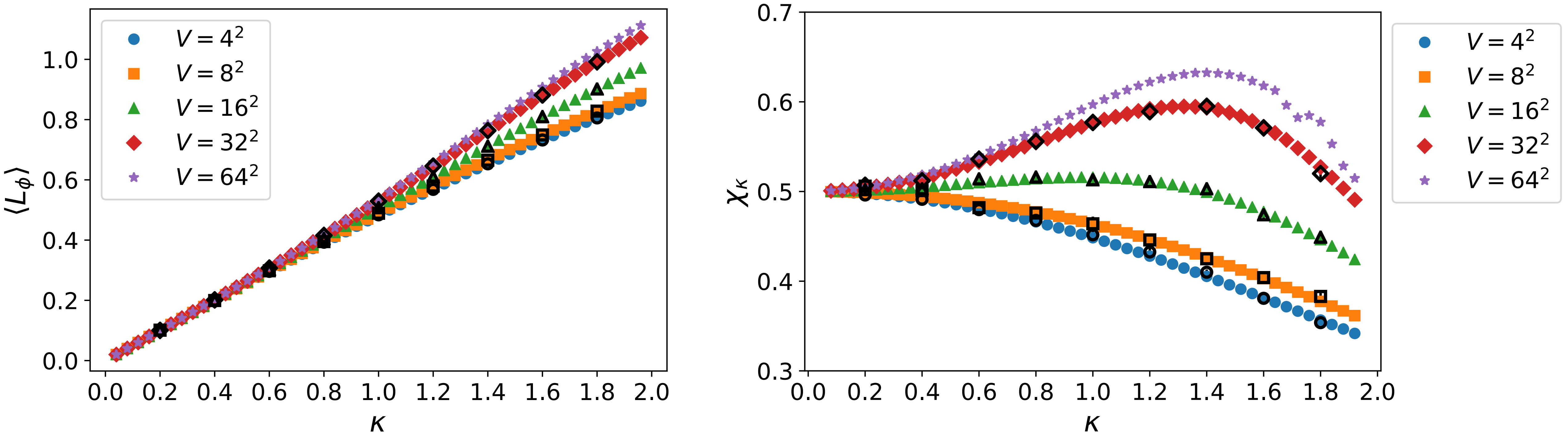
For $\beta = 0$, we can write exact value of the Polyakov loop in terms of Bessel functions. This provides a simple check of the tensor formulation. The exact expression is: $\langle P \rangle = 2\left(\frac{I_2(\kappa)}{I_1(\kappa)}\right)^{N_\tau}$,



Check: Comparison with MC

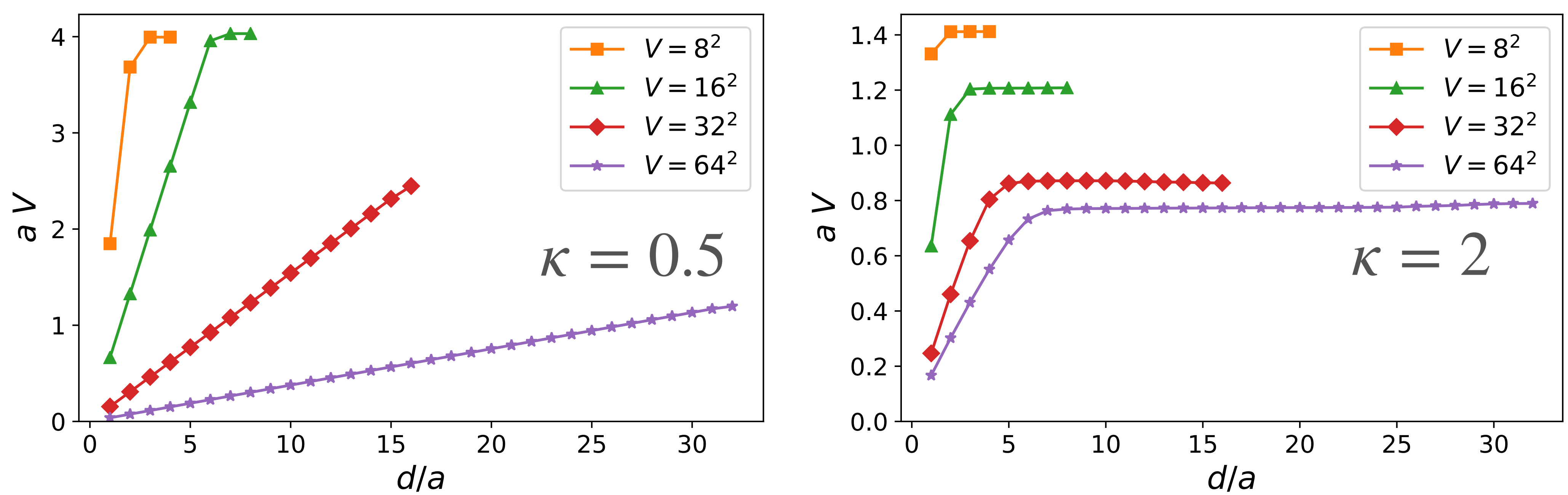
We compute some observables to compare the tensor network results to MC. On the left we show $\langle L_\phi \rangle = \frac{1}{V}\frac{-\partial \ln Z}{\partial \kappa} = \sum_{x,\mu} \mathrm{TrU}_{x,\mu}$ and on the right the second derivative. We also measure average plaquette and see agreement with MC results [black markers].



Results

Polyakov loop correlator is given by, $C(r) = \exp(-\beta V(r))$. This also provides a measure of monitoring confinement when $V \propto r$ (the slope gives σ, string tension). In a Higgs phase, it is constant and independent of r.

We show the results for correlator in two phases separated by cross-over at around $\kappa \approx 1.4$. The left shows the confining phase while for the one on the right a string breaking occurs and it goes to Higgs phase quickly for large lattices. Results are consistent with MC study done recently in 1402.7124



Future directions

- Two-dimensional Principal Chiral model [No tensor network study yet!]
- Computing Witten index from Z for supersymmetric gauge theories
- Classical Ising Model with magnetic field and exact form for canonical partition function "Z".
- Understanding the entanglement entropy for 2d NAGH model. HOTRG algorithms have been used to compute Renyi/Von Neumann entropy for several systems (see 1703.10577)
- Formulating tensor network for SU(N) gauge/Higgs system for 2d with $N > 2$?

THANK YOU!