Supersymmetry on the Lattice

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arXiv:1405.0644, arXiv:1410.6971, arXiv:1411.0166
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Context: Why lattice supersymmetry

Lattice discretization provides non-perturbative,
gauge-invariant regularization of gauge theories

We've discussed (in previous talks ?) many ways lattice studies can improve our knowledge of strongly coupled field theories

We can imagine many potential susy applications, including

- Compute Wilson loops, spectrum, scaling dimensions, etc., complementing perturbation theory, holography, bootstrap, ...
- Further direct checks of conjectured dualities
- Validate or refine AdS/CFT-based modelling (e.g., QCD phase diagram, condensed matter systems)

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  complementing perturbation theory, holography, bootstrap, ...
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  (e.g., QCD phase diagram, condensed matter systems)

Many ideas probably infeasible ; relatively few have been explored.

Context: Why not lattice supersymmetry

There is a problem with supersymmetry in discrete space-time

Recall supersymmetry extends Poincaré symmetry
by spinorial generators Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, \mathcal{N}$

The resulting algebra includes $\left\{Q_{\alpha}, \overline{Q}_{\dot{\alpha}}\right\} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu}$

P_{μ} generates infinitesimal translations, which don't exist on the lattice
$\Longrightarrow$ supersymmetry explicitly broken at classical level

Explicitly broken supersymmetry $\Longrightarrow$ relevant susy-violating operators
(typically many)

Fine-tuning their couplings to restore supersymmetry
is generally not practical in numerical lattice calculations

Special supersymmetric theories

There are certain theories where we can exactly preserve a subset of SUSY algebra based on the idea of twisting.

Maximal ($\mathcal{N} = 4$) supersymmetric Yang–Mills (SYM)

The only known 4d system with a supersymmetric lattice formulation

Remainder of talk will focus on recent progress with lattice $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is a particularly interesting theory

—Context for development of AdS/CFT correspondence

—Important for studies of Quark-Gluon Plasma (QGP) at strong couplings

—Arguably simplest non-trivial field theory in four dimensions

Exact susy on the lattice: $\mathcal{N} = 4$ SYM

Basic features:

- SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ},
  all massless and in adjoint rep.
- Action consists of kinetic, Yukawa and four-scalar terms
  with coefficients related by symmetries
- Supersymmetric: 16 supercharges Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, 4$
  Fields and Q's transform under global SU(4) $\simeq$ SO(6) R symmetry
- Conformal: β function is zero for any 't Hooft coupling λ

Twisted $\mathcal{N} = 4$ SYM

Everything transforms with **integer spin** under $\mathrm{SO}(4)_{tw}$ — **no spinors**

$$Q^{\mathrm{I}}_{\alpha} \text{ and } \overline{Q}^{\mathrm{I}}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$
$$\Psi^{\mathrm{I}} \text{ and } \overline{\Psi}^{\mathrm{I}} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$
$$A_\mu \text{ and } \Phi^{\mathrm{IJ}} \longrightarrow \mathcal{A}_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi}) \text{ and } \overline{\mathcal{A}}_a$$

The twisted-scalar supersymmetry $\mathcal{Q}$ acts as

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$
$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$
$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

1. $\mathcal{Q}$ directly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f.
2. The susy subalgebra $\mathcal{Q}^2 \cdot = 0$ is manifest

Lattice $\mathcal{N} = 4$ SYM

The lattice theory is very nearly a direct transcription

- Covariant derivatives $\longrightarrow$ finite difference operators
- Gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

- Naive lattice action retains same form as continuum action and remains supersymmetric, $\mathcal{Q}S = 0$

Geometrical formulation facilitates discretization

η live on lattice sites $\qquad$ ψ_a live on links

χ_{ab} connect opposite corners of oriented plaquettes

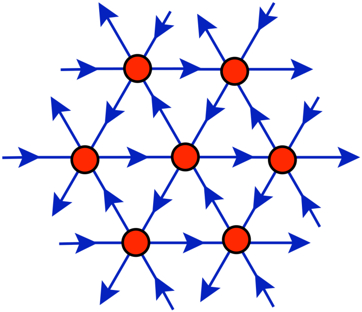
Orbifolding / dimensional deconstruction produces same lattice system

Five links in four dimensions $\longrightarrow A_4^*$ lattice

—Can picture A_4^* lattice
as 4d analog of 2d triangular lattice

—Preserves S_5 point group symmetry

—Basis vectors are non-orthogonal
and linearly dependent



S_5 irreps precisely match onto irreps of twisted $\text{SO}(4)_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1}: \quad \mathcal{U}_a \longrightarrow A_\mu + iB_\mu, \quad \phi + i\overline{\phi}$$
$$\psi_a \longrightarrow \psi_\mu, \quad \overline{\eta}$$
$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4}: \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \quad \overline{\psi}_\mu$$

Twisted $\mathcal{N} = 4$ SYM on the A_4^* lattice

—We have exact gauge invariance

—We exactly preserve $\mathcal{Q}$, one of 16 supersymmetries

—The S_5 point group symmetry
provides twisted R & Lorentz symmetry in the continuum limit

The high degree of symmetry has important consequences

- Moduli space preserved to all orders of lattice perturbation theory
  $\longrightarrow$ no scalar potential induced by radiative corrections
- β function vanishes at one loop in lattice perturbation theory
- Real-space RG blocking transformations preserve $\mathcal{Q}$ and S_5
- Only one marginal tuning to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum

The theory is **almost** suitable for practical numerical calculations. . .

New development (arXiv:1505.03135)

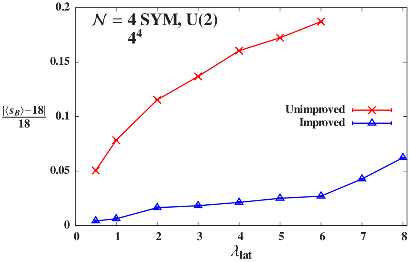
Scalar potential softly breaks $\mathcal{Q}$ supersymmetry

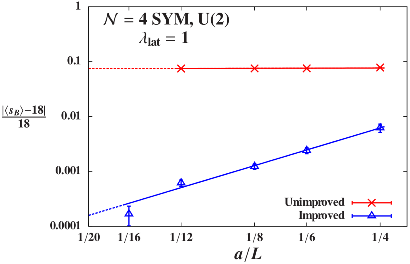
Plaquette determinant can be made $\mathcal{Q}$-invariant

Basic idea: Modify the equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \longrightarrow \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\sum_{a\neq b}\left[\det \mathcal{P}_{ab}(n) - 1\right]$$

Produces much smaller violations of $\mathcal{Q}$ Ward identity $\langle s_B\rangle = 9N^2/2$





Aside: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$
\begin{aligned}
S_{imp} &= S'_{exact} + S_{closed} + S'_{soft} \qquad (3.10)\\
S'_{exact} &= \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) \\
&\qquad + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det \mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\Bigg] - S_{det} \\
S_{det} &= \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det \mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{b})\psi_a(n+\widehat{b})\right] \\
S_{closed} &= -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right], \\
S'_{soft} &= \frac{N}{2\lambda_{\text{lat}}}\mu^2 \sum_n \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2
\end{aligned}
$$

The lattice action is obviously very complicated

(For experts: $\gtrsim$100 inter-node data transfers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971

Physics result: Static potential is Coulombic at all λ

Static potential $V(r)$ from $r \times T$ Wilson loops: $W(r,T) \propto e^{-V(r)\,T}$

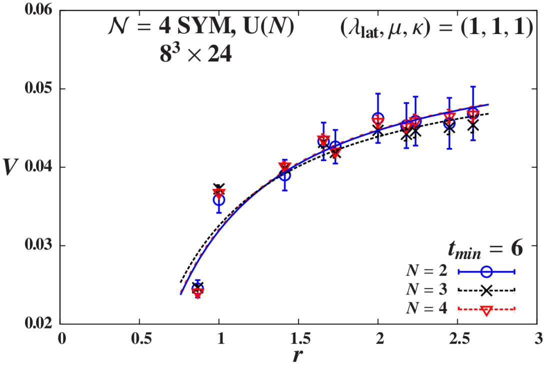
Fit $V(r)$ to Coulombic
or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$

C is Coulomb coefficient

σ is string tension



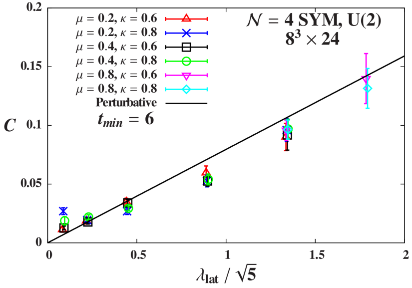
Fits to confining form always produce vanishing string tension $\sigma = 0$

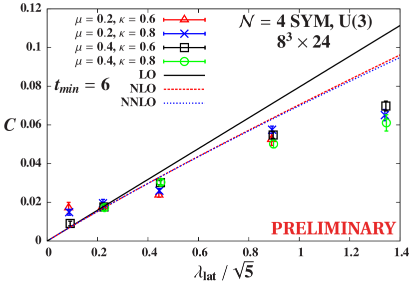
To be revisited with the improved action

Coupling dependence of Coulomb coefficient

Perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

AdS/CFT predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$, $\lambda \to \infty$, $\lambda \ll N$





Left: Agreement with perturbation theory for $N = 2$, $\lambda \lesssim 2$

Right: Tantalizing $\sqrt{\lambda}$-like discrepancy for $N = 3$, $\lambda \gtrsim 1$

No visible dependence on (unimproved) soft $\mathcal{Q}$ breaking

Sixteen supercharge supersymmetric QM

SYM consisting of sixteen supercharges in 1d at low temperatures with large N is conjectured to be dual to a black hole with N units of charge at same temperature. Energy of the black hole has been computed in SUGRA :

$$\epsilon \sim 7.41 N^2 t^{14/5}$$

with $\epsilon = E/\lambda^{1/3}$ and $t = T/\lambda^{1/3}$.

This has been checked on the SYM side with great success upto first order corrections in $\alpha^{'}$. In fact, leading order in $\alpha^{'}$ is a prediction of lattice simulations.

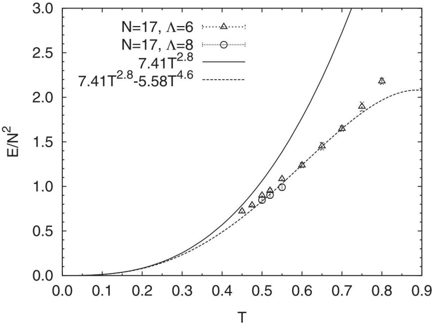
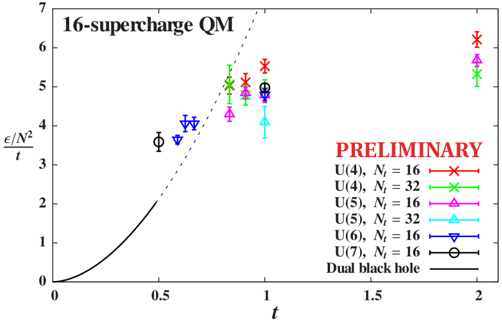


Figure: *Masanori Hanada, Yoshifumi Hyakutake, Jun Nishimura, and Shingo Takeuchi, Phys. Rev. Lett.* ***102****, 191602 (2009).*

Leading order behavior was also confirmed using lattice methods by S.Catterall and T.Wiseman (Phys. Rev. **D78**, 041502 (2008). [arXiv:0803.4273 [hep-th]])

Polyakov line in 1d case is non-vanishing even at low T, no phase transition in 1d, as predicted by the gauge/gravity correspondence. There is only single deconfined phase.

Revisiting p=0 with improved action and public code



$\mathcal{N} = (8,8)$ SYM in two dimensions

Unlike 1-d case discussed before, the maximally supersymmetric theory in two dimensions has a deconfinement/confinement phase transition. It is conjectured to be related to a different phase transition between black hole/black string in the dual supergravity theory.

We construct dimensionless coupling given by $\hat{\lambda} = \lambda\beta^2$, where $\beta = aN_t$. Other dimensionless quantities related to size of spatial and temporal directions can be defined as :

$$r_x = \sqrt{\lambda}R \quad \text{and} \quad r_\tau = \sqrt{\lambda}\beta$$

The energy power law from supergravity calculations is predicted :

$$\epsilon \sim N^2 t^3 \sqrt{\lambda} R \quad \forall \quad t \ll 1$$

with, $\epsilon = E/\sqrt{\lambda}, t = T/\lambda^{1/2}$ defined as the dimensionless energy and temperature respectively.

Recapitulation

- Lattice supersymmetry is both enticing and challenging

- $\mathcal{N} = 4$ SYM is practical to study on the lattice
  thanks to exact preservation of susy subalgebra $\mathcal{Q}^2 = 0$

- The theory is simple; the lattice action is complicated
  $\longrightarrow$ Public code to reduce barriers to entry

- The static potential is always Coulombic
  For $N = 2$ $C(\lambda)$ is consistent with perturbation theory
  For $N = 3$ we may be seeing behavior predicted by AdS/CFT

- Many more directions are being — or can be — pursued
  - $\mathcal{N} = 4$ anomalous dimensions, e.g. for Konishi operator
  - Understanding the (absence of a) sign problem
  - Systems with less supersymmetry, in lower dimensions, including matter fields, exhibiting spontaneous susy breaking, ...

Numerical complications

1. Complex gauge field $\Longrightarrow$ U(N) = SU(N) $\otimes$ U(1) gauge invariance
   U(1) sector decouples only in continuum limit

2. $\mathcal{Q}\,\mathcal{U}_a = \psi_a \Longrightarrow$ gauge links must be elements of algebra
   Resulting flat directions required by supersymmetric construction
   but must be lifted to ensure $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

We need to add two deformations to regulate flat directions

SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$

U(1) plaquette determinant $\sim G \sum_{a \neq b} \left(\det \mathcal{P}_{ab} - 1\right)$

Scalar potential **softly** breaks $\mathcal{Q}$ supersymmetry

↖susy-violating operators vanish as $\mu^2 \to 0$

Plaquette determinant can be made $\mathcal{Q}$-invariant (new development)

Interesting open problem - Free energy of $\mathcal{N} = 4$ SYM

The free energy was calculated at strong coupling using AdS/CFT correspondence in [Gubser et. al, Phys.Rev. D54 (1996) 3915, hep-th/9602135]. It was suggested that the leading term in expansion of F has the form :

$$F = -f(g_{YM}^2 N)\frac{\pi^2}{6}N^2 VT^4$$

where, $f(g_{YM}^2 N)$ is (possibly!) a smooth function which interpolates between a weak coupling limit of 1 and a strong coupling limit of 3/4.

Through lattice, we can explore the behavior of the free energy at intermediate couplings which might be useful for determining the exact form of $f(g_{YM}^2 N)$.

Supplement: The (absence of a) sign problem

In lattice gauge theory we compute operator expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O}\, e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\, \text{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

$\text{pf}\, \mathcal{D} = |\text{pf}\, \mathcal{D}| e^{i\alpha}$ can be complex for lattice $\mathcal{N} = 4$ SYM
$\longrightarrow$ Complicates interpretation of $\left[e^{-S_B}\, \text{pf}\, \mathcal{D}\right]$ as Boltzmann weight

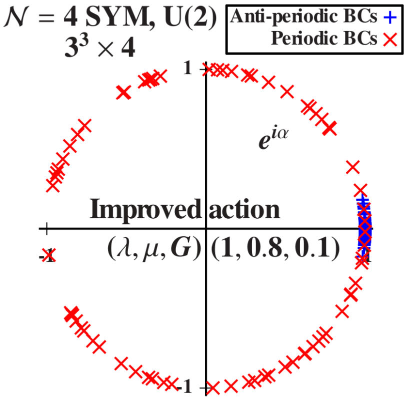
Have to **reweight** "phase-quenched" (pq) calculations

$$\langle \mathcal{O} \rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O}\, e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\, |\text{pf}\, \mathcal{D}| \qquad \langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq}}{\left\langle e^{i\alpha} \right\rangle_{pq}}$$

Sign problem: This breaks down if $\left\langle e^{i\alpha} \right\rangle_{pq}$ is consistent with zero

Illustration of sign problem and its absence

- With periodic temporal fermion boundary conditions
  we have an obvious sign problem, $\langle e^{i\alpha} \rangle_{pq}$ consistent with zero
- With anti-periodic BCs and all else the same $\langle e^{i\alpha} \rangle_{pq} \approx 1$
  $\longrightarrow$ phase reweighting not even necessary



Even stranger

Other $\langle \mathcal{O} \rangle_{pq}$ nearly identical
despite sign problem...

Can this be understood?

Pfaffian phase dependence on volume and N

No indication of a sign problem with anti-periodic BCs

- $1 - \langle\cos(\alpha)\rangle \ll 1$ means pf $\mathcal{D} = |\text{pf}\,\mathcal{D}|e^{i\alpha}$ nearly real and positive
- Fluctuations in pfaffian phase don't grow with the lattice volume
- Insensitive to number of colors $N = 2, 3, 4$
- To be revisited with the improved action

Hard calculations

Each $4^3 \times 6$ measurement required $\sim$8 days, $\sim$10GB memory

Parallel $\mathcal{O}(n^3)$ algorithm