Testing holography through lattice simulations

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Quantum Gravity, String theory and Holography
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Outline

1. Motivation

2. Lattice supersymmetry : Problems and resolutions

3. Topological twist and lattice action for $\mathcal{N} = 4$ SYM on A_4^* lattice

4. Holographic principle, 2d SYM and dual gravity predictions
   - Phase transition
   - Homogeneous D1 phase
   - Localized D0 phase

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Why lattice supersymmetry ?

Lattice discretization provides non-perturbative, gauge-invariant regularization of gauge theories

- Insights into confinement, symmetry breaking, conformal field theories , etc.
- Gauge/gravity (AdS/CFT) duality $\longrightarrow$ potential non-perturbative definition of string theory

Lattice studies are important for non-perturbative λ and finite-N regime of gauge theories.

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Lattice SUSY : Problems and resolutions

Problem

Supersymmetry generalizes Poincaré symmetry,
adding spinorial generators Q and $\overline{Q}$ to translations, rotations, boosts

The algebra includes $Q\overline{Q} + \overline{Q}Q = 2\sigma^\mu P_\mu$,

P_μ generates infinitesimal translations, which don't exist on the lattice
$\implies$ supersymmetry explicitly broken at the classical level.

Solution

Preserve a subset of SUSY algebra exactly on the lattice. Possible for theories with $Q \geq 2^D$. For ex : $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM). Methods are based on orbifold construction and twisiting. I will focus on the latter in this talk.

$\mathcal{N} = 4$ SYM is a particularly interesting theory

—Important for holographic dualities at large-N.

—Arguably simplest non-trivial field theory in four dimensions

Basic features:

- SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ}, all massless and in adjoint rep.
- Supersymmetric: 16 supercharges Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, 4$
  Fields and Q's transform under global $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$ R-symmetry
- Conformal: β function is zero for any 't Hooft coupling $\lambda = g^2_{YM} N$

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Topological twisting $\implies$ Exact SUSY on lattice

$$SO(4)_{tw} \equiv \text{diag}\left[SO(4)_{euc} \times SO(4)_R\right] \quad ; \quad SO(4)_R \subset SO(6)_R$$

The 16-real components of the spinors in $\mathcal{N} = 4$ SYM fill up the Dirac-Kähler multiplet :

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \gamma_a \mathcal{Q}_a + \gamma_a \gamma_b \mathcal{Q}_{ab}$$

$$\text{with } a, b = 1, \cdots, 5$$

$\mathcal{Q}$'s transform with integer spin under the "twisted rotation group".

This twisting and repackaging gives us a nilpotent, scalar supercharge $\mathcal{Q}$ which can be exactly preserved on the lattice even at finite lattice spacing.

Twisted $\mathcal{N} = 4$ SYM fields

Useful to start in a 5d setup

$$Q^{\mathrm{I}}_{\alpha} \text{ and } \overline{Q}^{\mathrm{I}}_{\dot{\alpha}} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab} \longrightarrow 1 + 5 + \overline{10}$$
$$\Psi^{\mathrm{I}} \text{ and } \overline{\Psi}^{\mathrm{I}} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$
$$A_\mu \text{ and } \Phi^{\mathrm{IJ}} \longrightarrow \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

Everything transforms with **integer spin** under $\mathrm{SO}(4)_{tw}$ — **no spinors**. Then under dimensional reduction :

$$\mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab} \longrightarrow \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\mathcal{A}_a \longrightarrow (A_\mu, \phi) + i(B_\mu, \overline{\phi})$$

where, a b runs from 1 ·· 5 and μ from 1 ·· 4

Lattice action

The lattice action consists of $\mathcal{Q}$-exact and $\mathcal{Q}$-closed terms.

$$S_{\text{lat}} = \frac{N}{4\lambda_{\text{lat}}} \mathcal{Q} \sum \left(\chi_{ab}\mathcal{F}_{ab} + \eta[\bar{\mathcal{D}}_a, \mathcal{D}_a] - \frac{1}{2}\eta d \right) - \frac{N}{16\lambda_{\text{lat}}} \sum \epsilon_{abcde}\chi_{ab}\, \bar{\mathcal{D}}_c\chi_{de}$$

- Covariant derivatives $\longrightarrow$ finite difference operators eg.

$$\mathcal{D}_a\psi_b \to \mathcal{U}_a(x)\psi_b(x+a) - \psi_b(x)\mathcal{U}_a(x+b)$$

and anti-symmetric field tensor defined as,

$$\mathcal{F}_{ab} = [\mathcal{D}_a, \mathcal{D}_b]$$

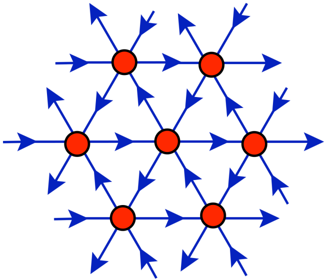
Catterall et al. 1405.0644

$\mathcal{Q}$ acts as :

$$\mathcal{Q}\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\psi_a = 0$$
$$\mathcal{Q}\chi_{ab} = -\bar{\mathcal{F}}_{ab} \qquad \mathcal{Q}\bar{\mathcal{A}}_a = 0$$
$$\mathcal{Q}\eta = d \qquad \mathcal{Q}d = 0$$

5d → 4d dimensional reduction gives A_4^* lattice

—Can think of A_4^* lattice
as 4d analog of 2d triangular lattice



—Preserves S_5 point group symmetry

— **Basis vectors are non-orthogonal**

S_5 irreps precisely match onto irreps of $\mathrm{SO}(4)_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1}: \quad \mathcal{U}_a \longrightarrow A_\mu + iB_\mu, \quad \phi + i\overline{\phi}$$

$$\psi_a \longrightarrow \psi_\mu, \quad \overline{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4}: \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \quad \overline{\psi}_\mu$$

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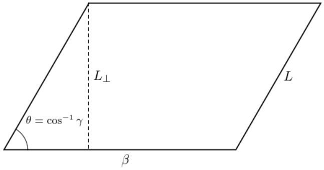

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Public code for maximal SYM

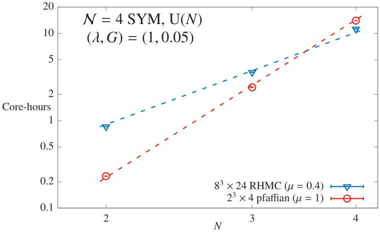
Our parallel code is publicly developed and available at :

github.com/daschaich/susy

We have recently generalized this for arbitrary N to access holographic dualities. The computational costs scales as : $N^{7/2}$

D. Schaich, RGJ et al. (In preparation, 2017).

This code evolved from MILC lattice QCD code and was first presented in arXiv:1410.6971 (restricted to $N \leq 4$).



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Holographic applications

Original AdS/CFT correspondence

4D $\mathcal{N} = 4$ U(N) super-Yang-Mills theory associated with N D3-branes, is dual to Type IIB string theory on $AdS_5 \times S_5$ in the large N limit.

More general holographic dualities in lower dimensions

Maximally superymmetric YM in $p + 1$ dimensions dual to Dp-branes
At low temperatures (not very low !) , and in the decoupling limit : dual description in terms of black holes in Type II A/B supergravity

Two-dimensional SYM (p=1)

- Dimensional reduction : $A_4^* \longrightarrow A_2^*$ giving a skewed torus with $\gamma = 1/2$ ($\gamma = \cos\theta$).
- 't Hooft coupling (λ) is dimensionful in two dimensions and we construct a dimensionless coupling given by $\hat{\lambda} = \lambda\beta^2$, where $\beta = 1/T$.
- Extent of spatial and time circles can be written as dimensionless quantities ; $r_x = \sqrt{\lambda}R$ and $r_\tau = \sqrt{\lambda}\beta = 1/t$, where t is the dimensionless temperature. In addition, we also have γ.
- Much more interesting than 1-d QM case, phase transition between uniform D1 and localized D0 phase with spatial Wilson line being the order parameter.
- Three interesting regimes : 1) $r_\tau \ll r_x$, 2) $r_\tau \sim r_x > 1$ and, 3) $r_\tau \gg r_x$

Regime of valid supergravity description

To have a valid SUGRA description, we need :

- Radius of curvature should be large in units of α'. This implies $r_\tau \gg 1$.
- String coupling should be small.

We can combine both requirements to get a constraint on the effective dimensionless coupling we can probe for a well-defined SUGRA description (p $<$ 3)

$$1 \ll \lambda_p \beta^{3-p} \ll N^{\frac{10-2p}{7-p}}$$

This can be written in terms of our dimensionless coupling for p=1 as,

$$1 \ll r_\tau \ll N^{\frac{2}{3}}$$

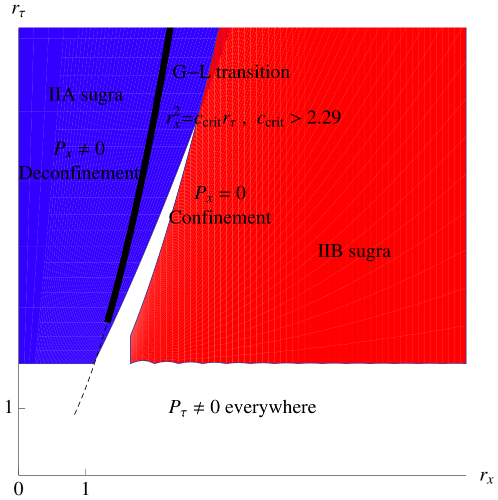

Figure 1: Different regions of the phase space for $\gamma = 0$. From : arXiv:1008.4964

Phase transition at large-N

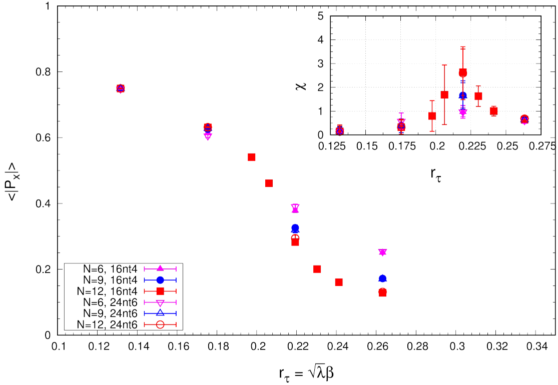

Figure 2: Phase transition for $\alpha = r_x/r_\tau = 4$. With, $\chi = N^2(\langle P_x^2\rangle - \langle P_x\rangle^2)$

$r_\tau \ll r_x$

- High temperature (weak coupling) regime of the theory, well approximated by BQM (Bosonic Quantum mechanics).
- Higher order phase transition(s) expected around : $\underline{r_x^3 \sim 2.09 r_\tau}$ (Note that this is $r_x^3 \sim 1.35 r_\tau$ for $\gamma = 0$)
- Bosonic action density $\sim t^2$.

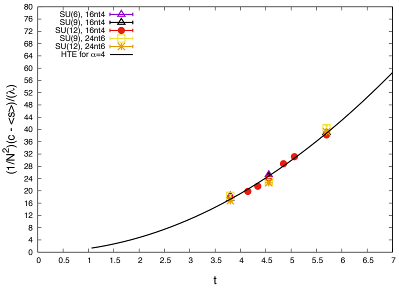

Figure 3: High-temperature lattice results

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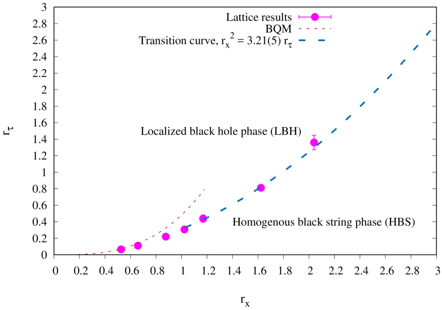
$r_\tau \sim r_x > 1$

- For the skewed torus, the gravity results are not yet available and our current lattice simulations predict phase transition along : $r_x^2 = 3.21(5)\ r_\tau$.



- Earlier work ($\gamma = 0$) predicted the transition along $r_x^2 \approx 2.45 r_\tau$ which has been confirmed from calculations on the gravity side two months back [1702.07718].

D1 thermodynamics

The bosonic action 'density' for the dual D1 branes is found to be independent of angle of skewness of the torus. In this homogeneous phase, it predicts, like the rectangular torus :

$$\frac{s_{\mathrm{B}os,\mathrm{h}omog}}{N^2\lambda} = -\frac{2^3\pi^{\frac{5}{2}}}{3^4}t^3 \approx -1.728t^3$$

for the SYM bosonic action density, with $t = 1/r_\tau$ the dimensionless temperature.

Our lattice simulations seem consistent with this prediction.

D1 thermodynamics - Results from lattice simulations

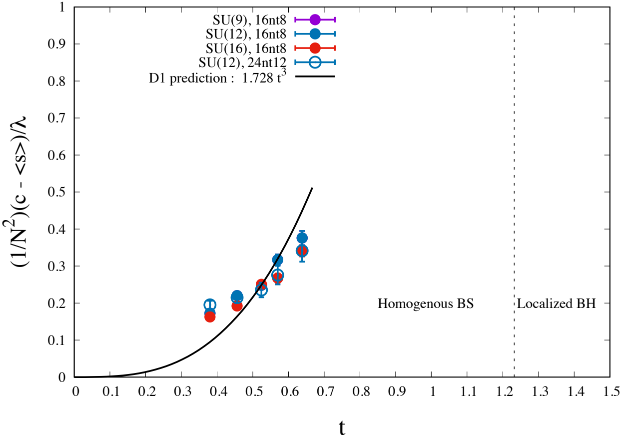


Figure 4: The lattice results and dual gravity prediction for $\alpha = 2$.

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$r_\tau \gg r_x$: Localised phase (D0 thermodynamics)

For the localized D0 phase, the prediction for the bosonic action density depends on the skewness of the torus and follows a different temperature-dependence than the D1 prediction

$$\frac{s_{\text{Bos},localized}}{N^2\lambda} \simeq -2.469 \frac{t^{\frac{16}{5}}}{\alpha^{\frac{2}{5}} \left(1-\gamma^2\right)^{\frac{7}{5}}}.$$

α is the aspect ratio defined as r_x/r_τ.

D0 thermodynamics - Results

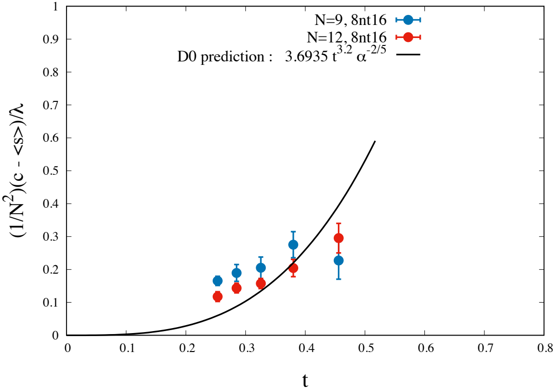


Figure 5: The lattice results and dual gravity prediction for the localized phase, $\alpha = 1/2$

Brief recap and future work

- Lattice supersymmetry is both enticing and challenging
- $\mathcal{N} = 4$ SYM is practical to study on the lattice
  thanks to exact preservation of susy subalgebra $\mathcal{Q}^2 = 0$
- We can now access relatively large-N to explore conjectured holographic dualities.
- We hope to study 3d SYM with sixteen supercharges in the future for which lattice construction is again possible.
- Computing the $\sqrt{\lambda}$ dependence of the static potential in $\mathcal{N} = 4$ SYM is also being pursued.

Thank you !

Thank you !

Funding and computing resources

The sign problem

In lattice gauge theory we compute operator expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O}\, e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\ \mathrm{pf}\,\mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

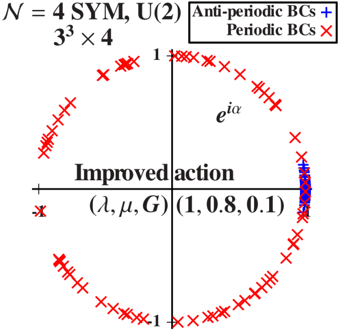
$\mathrm{pf}\,\mathcal{D} = |\mathrm{pf}\,\mathcal{D}|e^{i\alpha}$ can be complex for lattice $\mathcal{N} = 4$ SYM
$\longrightarrow$ Complicates interpretation of $\left[e^{-S_B}\ \mathrm{pf}\,\mathcal{D}\right]$ as Boltzmann weight

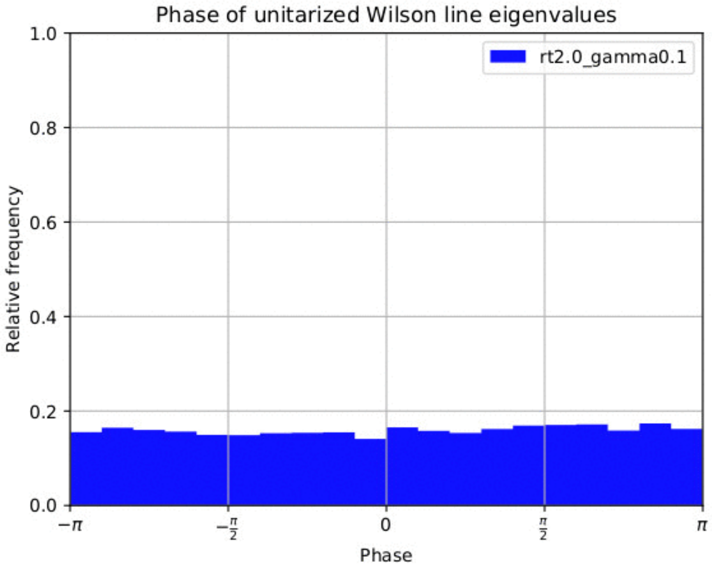
Sign problem: The complex phase produces a 'sign problem' if $\left\langle e^{i\alpha} \right\rangle$ is consistent with zero

Illustration - sign (problem, but no problem)

We have carefully analyzed the sign of the Pfaffian in this theory and find no evidence of it being a problem.

- With periodic temporal fermion boundary conditions
  we have an obvious sign problem, $\langle e^{i\alpha} \rangle$ consistent with zero
- With anti-periodic BCs and all else the same $\langle e^{i\alpha} \rangle \approx 1$





Phase of unitarized Wilson line eigenvalues
rt2.0_gamma0.1
Relative frequency
0.0
0.2
0.4
0.6
0.8
1.0
−π
−π/2
0
π/2
π
Phase

Truncated lattice theory

A naïve truncation of U(N) supersymmetric theory to SU(N) does not work.

- To maintain SU(N) gauge invariance it is necessary to keep the fermions in $\mathfrak{gl}(N, \mathbb{C})$, explicitly breaking the lattice supersymmetry that relates $\mathcal{U}_a$ to ψ_a in the U(N) construction.
- Solution : Represent the truncated gauge links as $\mathcal{U}_b = e^{iga\mathcal{A}_b}$ to argue that the continuum supersymmetry relating $\mathcal{A}_a$ and ψ_a is approximately realized in the large-N limit even at non-zero lattice spacing since $g \to 0$ in the decoupling limit.

Continuum vs. lattice coupling

The non-orthogonal basis vectors of the A_2^* lattice leads to mismatch in ' t Hooft coupling between lattice and continuum. The target continuum 2d-SYM coupling ($r_{\tau,cont.}$) differs from the lattice coupling as,

$$r_{\tau,lattice} = \sqrt{3\sqrt{3}}\; r_{\tau,cont.}$$